Two-Color Rado Numbers for Equal-Coefficient Non-Homogeneous Equations

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1 Abstract

A two-color Rado number asks how far one must color the positive integers before a monochromatic solution to a given equation is unavoidable. In this paper we study the equal-coefficient non-homogeneous equation

$$tx_1 + tx_2 + \cdots + tx_m = x_{m+1} + c, \qquad (E_{m,t,c})$$

where $m \geq 2$, $t \geq 1$, and $c \leq mt - 2$. Write

$$S = mt, \qquad D = S - 1 - c.$$

If S and c are both odd, the odd-even coloring shows that the Rado number is infinite. In every remaining parity case we determine the Rado number exactly:

$$R_2(E_{m,t,c}) = 1 + (mt - 1 - c)(mt^2 + t + 1).$$

The lower bound comes from an explicit three-block coloring. For the upper bound, we shift the positive integers to start at 0. A minimal-counterexample scaling argument reduces the problem to one constant c for a given pair of positive integers m and t.

The primitive problem becomes a question about sums of m same-colored integers. An equality between a red m-sum and a blue m-sum forces the same output to have both colors. A case analysis shows that this is unavoidable.

2 Introduction

Keywords: Ramsey Theory; Rado Numbers; Colorings; Linear Equations; Schur Equations

Def 2.1. A *two-coloring* of $[n] = \{1, 2, \ldots, n\}$ assigns each integer either red or blue. A solution of an equation is *monochromatic* if all of its variables receive the same color.

Def 2.2. The two-color Rado number $R_2(E)$ of an equation E is the least N such that every two-coloring of $[N]$ contains a monochromatic solution of E. If no such N exists, then $R_2(E) = \infty$.

The classical example is Schur's equation

$$x_1 + x_2 = x_3,$$

whose two-color Rado number is 5. This means that [4] has a monochromatic-solution-free coloring, but every two-coloring of [5] contains a monochromatic solution.

We focus on

$$t(x_1 + x_2 + \cdots + x_m) = x_{m+1} + c.$$

The sum of the input coefficients is

$$S = mt.$$

There is one immediate parity contradiction. If S and c are both odd, coloring odd numbers red and even numbers blue prevents a monochromatic solution. We therefore restrict attention to

$$Sc \equiv 0 \pmod 2, \qquad c \leq S - 2.$$

Several cases of this problem were already known exactly and can be used to check against the formula proved in this paper.

Case	Exact result	Source
$c = 0$, arbitrary m, t	$R_2 = N$	Guo–Sun
$t = 1$, arbitrary m, c	$R_2 = N$	Schaal

The main result of this writeup is that the same formula is exact for all equal coefficients in the acceptable parity range. In Section 3 we prove the lower bound. Section 4 shows how the formula connects with known results. In Section 5 we pass to nonnegative variables and reduce a hypothetical counterexample to only one constant c for each pair (m, t). Section 6 proves the two primitive upper bounds by an argument that compares the sum of m blue numbers to the sum of m red numbers and derives the full theorem. Section 7 gives conclusions and open problems.

3 The Three-Block Lower Bound

Our notation makes the lower bound simple.

Theorem 3.1. *Let $m \geq 2$ and $t \geq 1$, and let $c \leq mt - 2$. Put*

$$S = mt, \qquad D = S - 1 - c.$$

Then

$$R_2(E_{m,t,c}) \geq N := 1 + D(mt^2 + t + 1).$$

Equivalently,

$$N = m^2t^3 - mt^2c - tc + (m-1)t - c.$$

Proof. Color $[1, N-1]$ in three blocks:

red	blue	red
$[1, D]$	$[D+1, D(S+1)]$	$[D(S+1)+1, N-1]$.

We show that no monochromatic solution is possible.

Suppose first that all input variables are blue. Then every $x_i \geq D + 1$, so

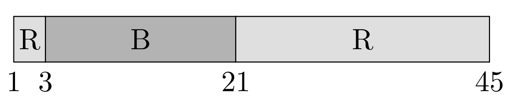
$$t(x_1 + \cdots + x_m) \geq tm(D+1) = S(D+1).$$

Hence
$$x_{m+1} \geq S(D+1) - c = D(S+1) + 1,$$
which is outside the blue block.

Suppose next that all input variables are red and lie in the lower red block. Then
$$S \leq t(x_1 + \cdots + x_m) \leq SD.$$
Therefore
$$D + 1 = S - c \leq x_{m+1} \leq SD - c \leq D(S+1),$$
so x_{m+1} is blue.

Finally suppose that all inputs are red and at least one input lies in the upper red block. Then for some j,
$$x_j \geq D(S+1) + 1.$$
The other $m-1$ inputs are at least 1, so
$$\begin{aligned} t(x_1 + \cdots + x_m) &\geq t\big(D(S+1)+1\big) + t(m-1) \\ &= S + tD(S+1). \end{aligned}$$
Thus
$$x_{m+1} \geq S + tD(S+1) - c = 1 + D\big(t(S+1)+1\big) = N,$$
which is outside the colored interval. All possibilities lead to a contradiction. $\square$

Example 3.2. Consider
$$2x_1 + 2x_2 + 2x_3 = x_4 + 2.$$
Here $m = 3$, $t = 2$, $S = 6$, and $D = 3$. The theorem gives
$$N = 1 + 3(3 \cdot 4 + 2 + 1) = 46.$$
A solution-free coloring of [45] is
$$[1,3] \text{ red}, \qquad [4,21] \text{ blue}, \qquad [22,45] \text{ red}.$$



4 How the Formula Fits Known Results

The same expression N recovers all three known equal-coefficient families.

4.1 The homogeneous case $c = 0$

When $c = 0$,
$$D = mt - 1.$$
The lower bound becomes
$$N = 1 + (mt-1)(mt^2 + t + 1) = m^2t^3 + (m-1)t.$$
Since $S = mt$, this is
$$tS^2 + S - t,$$
which is the exact Guo–Sun value.

4.2 Unit coefficients $t = 1$

When $t = 1$, the equation is

$$x_1 + \cdots + x_m = x_{m+1} + c.$$

Schaal's results give equality with N in the range considered here.

4.3 Two inputs $m = 2$

For

$$tx_1 + tx_2 = x_3 + c,$$

the formula becomes

$$N = 4t^3 - 2t^2c - tc + t - c.$$

This is also a special case of the general theorem proved in Section 6.

5 The Nonnegative Formulation

The upper bound becomes cleaner after shifting the positive integers so that 0 is available. Recall that

$$D = mt - 1 - c > 0.$$

Set

$$x_i = X_i + 1 \quad (1 \le i \le m), \qquad x_{m+1} = Z + 1.$$

Then $E_{m,t,c}$ is equivalent to

$$t(X_1 + \cdots + X_m) + D = Z. \tag{$F_{m,t,D}$}$$

All variables in $F_{m,t,D}$ are nonnegative integers.

Def 5.1. Let $r_{m,t}(D)$ be the least L such that every two-coloring of $\{0, 1, \ldots, L\}$ contains a monochromatic solution to $F_{m,t,D}$. If no such L exists, set $r_{m,t}(D) = \infty$.

The shift is a bijection between colorings and between solutions, so

$$R_2(E_{m,t,c}) = 1 + r_{m,t}(D). \tag{5.1}$$

In particular, the lower bound from Section 3 is equivalent to

$$r_{m,t}(D) \ge D(mt^2 + t + 1).$$

For convenience put

$$C = mt^2 + t + 1.$$

We will prove the following shifted form of the main theorem.

Theorem 5.2 (Nonnegative equal-coefficient theorem)**.** *Let $m \ge 2$, $t \ge 1$, and $D \ge 1$. If mtD is odd, then*

$$r_{m,t}(D) = \infty.$$

If mtD is even, then

$$r_{m,t}(D) = DC.$$

The lower bound has already been proved, so the substance is the upper bound. We begin by reducing a hypothetical counterexample to only $D = 1$ and $D = 2$.

5.1 Reduction

Fix m and t, and suppose for contradiction that the asserted upper bound fails for at least one positive D with mtD even. Choose such a D as small as possible. Thus there is a two-coloring

$$\chi : \{0, 1, \ldots, DC\} \longrightarrow \{\text{blue}, \text{red}\}$$

containing no monochromatic solution to $F_{m,t,D}$.

Lemma 5.3. *If mt is even, then the minimal counterexample has $D = 1$.*

Proof. Suppose mt is even and $D > 1$. By minimality of D, every two-coloring of $[0, C]$ contains a monochromatic solution to

$$t(y_1 + \cdots + y_m) + 1 = y_{m+1}.$$

Define a two-coloring χ' of $[0, C]$ by

$$\chi'(u) = \chi(Du).$$

Choose a monochromatic solution

$$t(u_1 + \cdots + u_m) + 1 = u_{m+1}$$

for χ'. Set $x_i = Du_i$ for every i. Then all x_i have the same color under χ, they lie in $[0, DC]$, and multiplying the displayed equation by D gives

$$t(x_1 + \cdots + x_m) + D = x_{m+1}.$$

This is a monochromatic solution for χ, a contradiction. □

Lemma 5.4. *If mt is odd, then the minimal counterexample has $D = 2$.*

Proof. Suppose mt is odd. Since mtD is even, D is even. If $D \neq 2$, then $D > 2$. By minimality of D, every two-coloring of $[0, 2C]$ contains a monochromatic solution to

$$t(y_1 + \cdots + y_m) + 2 = y_{m+1}.$$

Define

$$\chi'(v) = \chi\left(\frac{D}{2}\, v\right) \qquad (0 \leq v \leq 2C).$$

Choose a monochromatic solution

$$t(v_1 + \cdots + v_m) + 2 = v_{m+1}.$$

Set

$$x_i = \frac{D}{2}\, v_i.$$

Then $0 \leq x_i \leq DC$, all x_i have the same color under χ, and multiplying the displayed equation by $D/2$ gives

$$t(x_1 + \cdots + x_m) + D = x_{m+1},$$

again a contradiction. □

Thus only $D = 1$ when mt is even and $D = 2$ when mt is odd remain.

5.2 Two elementary sum lemmas

Interchange the two colors if necessary so that

$$\chi(0) = \text{blue}.$$

We will repeatedly use the following two observations.

Lemma 5.5. *Suppose $[0, DC]$ is colored red and blue with no monochromatic solution to $F_{m,t,D}$. If*

$$b_1 + \cdots + b_m = r_1 + \cdots + r_m = Q,$$

where every b_i is blue and every r_i is red, then

$$Q > D(mt + 1).$$

Proof. Suppose instead that $Q \leq D(mt + 1)$. Then

$$tQ + D \leq tD(mt + 1) + D = D(mt^2 + t + 1) = DC,$$

so $\chi(tQ + D)$ is defined. Both

$$(b_1, \ldots, b_m, tQ + D) \quad \text{and} \quad (r_1, \ldots, r_m, tQ + D)$$

solve $F_{m,t,D}$. If $tQ + D$ is blue, the first tuple is monochromatic; if it is red, the second tuple is monochromatic. Either possibility is a contradiction. □

Lemma 5.6 (Interval sums)**.** *Let $a \leq b$ be nonnegative integers. Every integer in*

$$[ma, mb]$$

can be written as the sum of m not necessarily distinct integers from $[a, b]$.

Proof. Start with m copies of a and increase the summands one at a time. This achieves every total from ma through mb. □

We call a representation of Q as a sum of m blue integers a *blue m-sum*, and define a red m-sum similarly.

6 The Primitive Upper Bounds

We now rule out both primitive counterexamples while keeping the least-colored-integer argument explicit.

6.1 Case 1: $D = 1$ and mt even

The equation is

$$t(x_1 + \cdots + x_m) + 1 = x_{m+1}.$$

The all-zero input forces 1 to be red. Using m copies of 1 then forces

$$H := mt + 1$$

to be blue. No positive integer at most H can be both a blue m-sum and a red m-sum.

Let v be the smallest positive blue integer. Then

$$2 \leq v \leq H.$$

Since $v = v + (m - 1) \cdot 0$ is a blue m-sum, it cannot also be a red m-sum.

Lemma 6.1. *In this case $v = 2$.*

Proof. Suppose $v > 2$. Then every integer in $[1, v-1]$ is red, so the interval-sum lemma implies that every integer in

$$[m, m(v-1)]$$

is a red m-sum. Since

$$m(v-1) \geq 2(v-1) \geq v,$$

the fact that v is not a red m-sum forces $v < m$.

Let k be the smallest positive integer such that $kv \geq m$. Then $k \leq m$ and

$$kv < m + v < 2m.$$

Hence

$$kv = k \cdot v + (m-k) \cdot 0$$

is a blue m-sum. Also,

$$m \leq kv < 2m \leq m(v-1),$$

so kv is a red m-sum. So we get

$$kv > mt + 1.$$

Since $kv < 2m$, we have

$$2m > mt + 1,$$

which forces $t = 1$.

Because mt is even, m is then even. Since $v > 2$, both 1 and 2 are red. The red inputs

$$2, 1, \ldots, 1$$

force $m + 2$ to be blue. On the other hand $H = m + 1$ and 0 are blue, so the blue inputs

$$H, 0, \ldots, 0$$

force $m + 2$ to be red. This contradiction proves $v = 2$. $\square$

Thus 0 and 2 are blue while 1 is red. If m is even, then

$$m = \underbrace{1 + \cdots + 1}_{m \text{ terms}}$$

is a red m-sum, while

$$m = \frac{m}{2} \cdot 2 + \frac{m}{2} \cdot 0$$

is a blue m-sum. Since $m \leq mt + 1$, we get a contradiction. Hence m is odd. Since mt is even, t is even.

Lemma 6.2. *Every odd integer at most $mt + 1$ is red.*

Proof. Suppose otherwise, and let $n \le mt+1$ be the smallest odd blue integer.

First suppose $n \le m$. Since m and n are odd,

$$m = n + \frac{m-n}{2} \cdot 2 + \frac{m+n-2}{2} \cdot 0$$

is a blue m-sum. But

$$m = \underbrace{1 + \cdots + 1}_{m \text{ terms}}$$

is a red m-sum. This is a contradiction because $m \le mt+1$.

Now suppose $n > m$. Every odd integer in $[1, n-2]$ is red. Sums of m such odd integers attain every odd integer from m through $m(n-2)$: indeed, write each odd summand as $1+2z$ and apply the interval-sum lemma to the z's. Since $n > m \ge 3$,

$$n \le m(n-2),$$

so n is a red m-sum. But

$$n = n + (m-1) \cdot 0$$

is a blue m-sum. Since $n \le mt+1$, this is again a contradiction. $\square$

Finally, $mt+1$ is odd because m is odd and t is even. The preceding lemma says that $mt+1$ is red, contradicting the fact that it was already forced blue. Therefore no primitive counterexample exists when $D = 1$.

6.2 Case 2: $D = 2$ and mt odd

Now m and t are both odd, and $m \ge 3$. The equation is

$$t(x_1 + \cdots + x_m) + 2 = x_{m+1}.$$

The all-zero input forces 2 to be red. Using m copies of 2 then forces

$$H := 2(mt+1)$$

to be blue. No positive integer at most H can be both a blue m-sum and a red m-sum.

Let $v \le H$ be the smallest blue positive integer greater than 2.

Lemma 6.3. *The integer* 1 *is blue.*

Proof. Suppose instead that 1 is red. Then every integer in $[1, v-1]$ is red, so every integer in

$$[m, m(v-1)]$$

is a red m-sum. The number

$$v = v + (m-1) \cdot 0$$

is a blue m-sum, and $v \le H$, so it cannot also be a red m-sum. Since

$$m(v-1) \ge 3(v-1) > v,$$

we must have $v < m$.

Let k be the smallest positive integer such that $kv \geq m$. Then $k \leq m$ and

$$kv < m + v < 2m \leq m(v-1).$$

Thus kv is a red m-sum, while

$$kv = k \cdot v + (m-k) \cdot 0$$

is a blue m-sum. Since kv is both a red and blue m-sum, we have

$$kv > 2(mt+1),$$

but $kv < 2m \leq 2(mt+1)$, a contradiction. Hence 1 is blue. $\square$

We now know that 0, 1, and v are blue, while every integer in $[2, v-1]$ is red. Since v is a blue m-sum and $v \leq H$, it cannot be a red m-sum. The interval-sum lemma fills

$$[2m, m(v-1)]$$

with red m-sums. Because

$$m(v-1) \geq 3(v-1) > v,$$

it follows that

$$v < 2m.$$

We split into three subcases.

Subcase 2a: $m < v < 2m$**.** The identity

$$2m = v + (2m-v) \cdot 1 + (v-m-1) \cdot 0$$

uses exactly m blue summands, so $2m$ is a blue m-sum. But

$$2m = \underbrace{2 + \cdots + 2}_{m \text{ terms}}$$

is a red m-sum. Since $2m \leq 2(mt+1) = H$, this is a contradiction.

Subcase 2b: $4 \leq v \leq m$**.** Let k be the smallest positive integer such that

$$kv \geq 2m.$$

Then $k \leq m$ and

$$kv < 2m + v \leq 3m.$$

Thus

$$kv = k \cdot v + (m-k) \cdot 0$$

is a blue m-sum. Since $v \geq 4$,

$$kv < 3m \leq m(v-1),$$

and hence $kv \in [2m, m(v-1)]$, so kv is also a red m-sum. We get

$$kv > 2(mt+1).$$

Because $kv < 3m$, we obtain

$$3m > 2(mt+1),$$

which forces $t = 1$.

Now $H = 2m + 2$. Also,

$$m(v-1) \geq 3m > 2m + 2 = H.$$

Therefore $H \in [2m, m(v-1)]$, so H is a red m-sum. But H itself is blue and 0 is blue, hence

$$H = H + (m-1) \cdot 0$$

is also a blue m-sum. This is a contradiction.

Subcase 2c: $v = 3$. Since m is odd,

$$2m = \frac{m+1}{2} \cdot 3 + \frac{m-3}{2} \cdot 1 + 1 \cdot 0$$

is a blue m-sum. It is also the red m-sum of m copies of 2. Again $2m \leq H$, a contradiction.

All possibilities are impossible. Therefore no primitive counterexample exists when $D = 2$.

6.3 Completion of the upper bound

We have shown that a minimal counterexample would have to be $D = 1$ when mt is even or $D = 2$ when mt is odd, and both primitive cases are impossible. Hence, whenever mtD is even,

$$r_{m,t}(D) \leq DC.$$

Together with the lower bound,

$$r_{m,t}(D) = D(mt^2 + t + 1).$$

If mtD is odd, then m, t, and D are all odd. The odd-even coloring of the nonnegative integers contains no monochromatic solution: all-even inputs give an odd output, while m odd inputs together with odd t and odd D give an even output. Thus

$$r_{m,t}(D) = \infty.$$

This completes the proof of the nonnegative theorem.

Returning to positive integers gives the main result.

Theorem 6.4 (Equal-coefficient Rado number)**.** *Let $m \geq 2$, $t \geq 1$, and $c \leq mt - 2$. If mt and c are not both odd, then*

$$R_2(E_{m,t,c}) = 1 + (mt - 1 - c)(mt^2 + t + 1).$$

Equivalently,

$$R_2(E_{m,t,c}) = m^2t^3 - mt^2c - tc + (m-1)t - c.$$

If mt and c are both odd, then $R_2(E_{m,t,c}) = \infty$.

Proof. Put $D = mt - 1 - c$. Equation (5.1) gives

$$R_2(E_{m,t,c}) = 1 + r_{m,t}(D).$$

If mt and c are both odd, then mtD is odd and the shifted Rado number is infinite. Otherwise mtD is even, so

$$R_2(E_{m,t,c}) = 1 + D(mt^2 + t + 1),$$

as claimed. □

As a special case, when $m = 2$ this gives

$$R_2(tx_1 + tx_2 = x_3 + c) = 4t^3 - 2t^2c - tc + t - c.$$

7 Conclusions and Open Problems

For $c \le mt - 2$, the equal-coefficient two-color problem is now completely determined. If mt and c are both odd, the odd-even coloring avoids a monochromatic solution forever. In every other parity case,

$$R_2(E_{m,t,c}) = 1 + (mt - 1 - c)(mt^2 + t + 1).$$

The proof has two main pieces: the three-block coloring gives the lower bound, and after shifting to nonnegative coordinates, one can further reduce the problem to the constants of 1 and 2, and finally a case analysis comparing sums of m numbers of the same color yields a contradiction.

The boundary value $c = mt - 1$ is also immediate: setting every variable equal to 1 gives a solution, so the Rado number is 1. The case $c \ge mt$ is different because the three-block construction used here no longer applies directly.

Open Problem 7.1. *Determine the two-color Rado number of*

$$t(x_1 + \cdots + x_m) = x_{m+1} + c$$

when $c \ge mt$.

Open Problem 7.2. *Determine the r-color Rado number of $E_{m,t,c}$ for $r \ge 3$.*

Open Problem 7.3. *Determine the two-color Rado number of*

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m = x_{m+1} + c,$$

where $a_1, a_2, \ldots, a_m$ are a fixed set of positive integers (not necessarily equal), and c is an integer constant.

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References

[1] S. Guo and Z.-W. Sun, *Determination of the two-color Rado number for $a_1x_1+\cdots+a_mx_m = x_0$*, Journal of Combinatorial Theory, Series A **115**(2) (2008), 345–353.

[2] B. Hopkins and D. Schaal, *On Rado numbers for $\sum_{i=1}^{m-1} a_ix_i = x_m$*, Advances in Applied Mathematics **35** (2005), 433–441.

[3] B. M. Landman and A. Robertson, *Ramsey Theory on the Integers*, 2nd ed., American Mathematical Society, 2014.

[4] R. Rado, *Studien zur Kombinatorik*, Mathematische Zeitschrift **36** (1933), 424–470.

[5] D. Schaal, *On generalized Schur numbers*, Congressus Numerantium **98** (1993), 178–187.