An Expository on 2-Color Rado Numbers

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July 28, 2026

1 Introduction

Ramsey Theory is the branch of combinatorics concerned with the emergence of order in large mathematical structures. At its core is the principle that complete disorder cannot persist indefinitely. Whenever a structure becomes sufficiently large, some form of regularity must inevitably appear. One of the earliest settings in which this phenomenon was studied is graph theory, where Ramsey's Theorem guarantees that sufficiently large edge-colored graphs must contain highly organized monochromatic subgraphs. This same philosophy has since been extended to many other areas of mathematics, including number theory, logic, and theoretical computer science.

One natural question arising from this philosophy is whether arithmetic exhibits the same unavoidable structure as graphs. Rather than searching for monochromatic subgraphs, one may ask whether every coloring of the positive integers must contain monochromatic solutions to certain linear equations. Richard Rado answered this question in 1933 by proving a remarkable characterization of exactly which homogeneous linear equations possess this property. His theorem not only unifies earlier results such as Schur's Theorem and Van der Waerden's Theorem but also serves as the foundation for the modern study of partition regularity and Rado numbers.

While Rado's Theorem determines whether monochromatic solutions must exist, it does not answer the quantitative question of how large a finite coloring must be before such a solution is guaranteed. This naturally leads to the notion of a *Rado number*, which measures the smallest interval for which every coloring necessarily contains a monochromatic solution to a given equation. Determining these numbers has become a central problem in arithmetic Ramsey Theory, combining techniques from combinatorics, number theory, and, more recently, computational methods.

The goal of this survey is to provide an accessible introduction to the development of Rado's Theorem and the theory of Rado numbers. We begin by tracing the historical progression from Ramsey's Theorem through the work of Schur and Van der Waerden, before introducing Rado's characterization of partition-regular equations. We then survey several important results on exact two-color Rado numbers for both homogeneous and non-homogeneous equations, concluding with a brief introduction to the use of SAT solvers as a modern computational tool for determining new Rado numbers. Throughout, we emphasize intuition alongside mathematical rigor by motivating new concepts, illustrating them with examples, and highlighting the key ideas behind the major results.

2 History

The road to Rado's Theorem was shaped by a sequence of remarkable discoveries that gradually extended the central philosophy of Ramsey Theory from graphs to arithmetic. What began as a question about unavoidable structure in edge-colored graphs has evolved into a broader investigation of whether similar phenomena occur among the positive integers.

In this section, we follow that historical progression. We begin with Ramsey's original theorem, which established the principle that sufficiently large graphs necessarily contain highly organized substructures. We then examine Schur's arithmetic analogue and Van der Waerden's theorem on arithmetic progressions, two landmark results demonstrating that unavoidable order also arises in number theory. These developments culminate in Rado's Theorem, which provides a complete characterization of the homogeneous linear equations that are guaranteed to admit monochromatic solutions under every finite coloring of the positive integers. Together, these results form the historical and mathematical foundation for the modern study of Rado numbers.

2.1 Ramsey's Theorem

Frank P. Ramsey proved Ramsey's Theorem in 1930 [1] while studying problems in mathematical logic. Although originally motivated by questions in logic, the theorem has become one of the cornerstones of modern combinatorics and serves as the guiding principle behind much of Ramsey Theory.

To understand the theorem, imagine coloring every edge of a graph either red or blue. At first glance, it may seem possible to avoid creating any large monochromatic pattern simply by coloring the edges randomly. Ramsey's Theorem shows that this intuition is false. Once the graph contains sufficiently many vertices, some highly organized monochromatic structure must appear regardless of how the edges are colored. We now state the theorem.

Theorem 2.1 (Ramsey's Theorem [1])**.** *For every pair of positive integers $r, s \in \mathbb{N}$, there exists a least positive integer $n = R(r, s)$ such that every 2-coloring (red and blue) of the edges of a graph K_n contains either a complete subgraph on r vertices, all of whose edges are blue, or a complete subgraph on s vertices, all of whose edges are red.*

Perhaps the most famous illustration of Ramsey's Theorem is the statement that among any six people, there must exist either three mutual acquaintances or three mutual strangers. Equivalently, this says that $R(3, 3) = 6$. Although this result is often presented as a puzzle, it beautifully illustrates the central philosophy of Ramsey Theory: sufficiently large structures necessarily contain highly organized patterns that cannot be avoided.

Although Ramsey's Theorem is formulated in the language of graph theory, its underlying philosophy extends far beyond graphs. This naturally raises the question of whether similar phenomena occur in arithmetic. Rather than searching for monochromatic subgraphs, can every coloring of the positive integers be forced to contain monochromatic solutions to a linear equation? Issai Schur provided the first major answer to this question in 1916.

2.2 Schur's Theorem

Schur's Theorem is now recognized as one of the foundational results of arithmetic Ramsey Theory. It demonstrates that the unavoidable order observed in graphs also appears in arithmetic. No matter how the positive integers are colored, one color class must eventually contain a solution to the equation

$$x + y = z.$$

We now state Schur's Theorem.

Theorem 2.2 (Schur's Theorem [2]). *For every finite coloring of the positive integers, there exist positive integers x, y, and z of the same color satisfying*

$$x + y = z.$$

Schur's Theorem marks an important shift in perspective. Instead of studying the structure of graphs, it studies the arithmetic structure of the positive integers. Regardless of how the integers are colored, it is impossible to avoid a monochromatic solution to $x+y = z$. This illustrates that the central philosophy of Ramsey Theory extends naturally from graph theory to arithmetic.

Before studying this theorem further, we introduce some terminology that will be used throughout the remainder of this paper.

Definition 2.3. Let

$$[n] = \{1, 2, \ldots, n\}.$$

An r**-coloring** of $[n]$ is a function

$$c : [n] \to [r],$$

where

$$[r] = \{1, 2, \ldots, r\}$$

represents a collection of r distinct colors.

In other words, an r-coloring assigns exactly one of r possible colors to every integer in the set $\{1, 2, \ldots, n\}$.

Example 2.4. Consider a 2-coloring of the set $\{1, 2, 3, 4\}$. One possible coloring is

$$\{1, 2, 3, 4\}.$$

Equivalently, this coloring may be viewed as the function

$$c : [4] \to [2].$$

Many problems in Ramsey Theory are concerned with finding collections of integers that all receive the same color. This motivates the following definition.

Definition 2.5. Let $c : [n] \to [r]$ be an r-coloring. A collection of integers

$$x_1, x_2, \ldots, x_k \in [n]$$

is said to be **monochromatic** if

$$c(x_1) = c(x_2) = \cdots = c(x_k).$$

For example, in the coloring above, the set $\{2, 4\}$ is monochromatic because both integers are colored blue, whereas the set $\{1, 2, 3\}$ is not monochromatic since its elements receive different colors.

Using this terminology, Schur's Theorem states that every finite coloring of the positive integers contains a monochromatic solution to the equation

$$x + y = z.$$

While Schur's Theorem guarantees that such a solution must eventually exist, it does not answer a natural quantitative question: *How many integers must be colored before a monochromatic solution is guaranteed?*

This question leads to one of the earliest Ramsey-type extremal functions.

Definition 2.6. For a positive integer r, the **Schur number**, denoted by $S(r)$, is the smallest positive integer S such that every r-coloring of

$$\{1, 2, \ldots, S\}$$

contains a monochromatic solution to

$$x + y = z.$$

Rather than merely asserting the existence of monochromatic solutions, Schur numbers measure precisely when such solutions become unavoidable. They are among the earliest examples of Ramsey-type extremal functions and illustrate a recurring theme throughout this survey. Once existence is established, the next challenge is determining the smallest finite structure in which that phenomenon must occur.

Despite more than a century of research, remarkably few Schur numbers are known exactly. At present,

$$S(1) = 2, \qquad S(2) = 5, \qquad S(3) = 14, \qquad S(4) = 45, \qquad S(5) = 161.$$

The computation of $S(5)$ required one of the largest computer-assisted proofs in combinatorics, generating more than two petabytes of proof data [3]. Determining $S(6)$ remains a major open problem. This remarkable achievement also highlights the increasingly important role that computational methods play in modern Ramsey Theory, a topic we return to later when discussing SAT solvers.

2.3 Van der Waerden's Theorem

Schur's Theorem demonstrates that monochromatic solutions to the equation

$$x + y = z$$

are unavoidable under every finite coloring of the positive integers. Do there exist other arithmetic patterns that must also appear in sufficiently large colorings? One of the most important answers was provided by Bartel van der Waerden in 1927 [4], who proved that monochromatic arithmetic progressions are likewise unavoidable. His theorem is a cornerstone of arithmetic Ramsey Theory and represents another major step toward the general framework later established by Rado.

Before stating the theorem, we introduce the notion of an arithmetic progression.

Definition 2.7. An **arithmetic progression** is a sequence of the form

$$a,\ a + d,\ a + 2d,\ \ldots,\ a + (k - 1)d,$$

where a is the initial term and $d > 0$ is a fixed integer called the *common difference*. Such a sequence is called a **k-term arithmetic progression**.

Arithmetic progressions are among the simplest and most fundamental patterns in number theory. Some examples include:

$$\{1, 2, 3\}, \{7, 14, 21\}, \text{ and } \{15, 30, 45, 60\},$$

whose common differences are 1, 7, and 15, respectively.

Van der Waerden's Theorem asserts that these patterns cannot be avoided once a finite coloring becomes sufficiently large.

Theorem 2.8 (Van der Waerden's Theorem [4]). *For every pair of positive integers r and k, there exists a least positive integer $W = W(r, k)$ such that every r-coloring of*

$$\{1, 2, \ldots, W\}$$

contains a monochromatic arithmetic progression of length k.

The integer $W = W(r, k)$ is called the **Van der Waerden number**. In this sense, Van der Waerden numbers play the same role for arithmetic progressions that Schur numbers play for the equation

$$x + y = z.$$

One of the smallest nontrivial examples is

Example 2.9.

$$W(2,3) = 9.$$

This result states that every 2-coloring of the integers

$$\{1, 2, \ldots, 9\}$$

must contain a monochromatic three-term arithmetic progression.

Example 2.10. Consider the following 2-coloring:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

The integers

$$\{1, 5, 9\}$$

form a monochromatic arithmetic progression with common difference 4. This example illustrates the type of structure that Van der Waerden's Theorem guarantees in every sufficiently large coloring.

Schur's Theorem guarantees monochromatic solutions to a particular linear equation, while Van der Waerden's Theorem guarantees monochromatic arithmetic progressions. Although these results appear different, they both reflect the same underlying principle. Sufficiently large colorings inevitably contain arithmetic structure. One might wonder which specific systems of linear equations are guaranteed to admit monochromatic solutions under every finite coloring of the positive integers.

Richard Rado answered this question in 1933 with his characterization of partition regular linear equations, now known as *Rado's Theorem*.

2.4 Rado's Theorem

The results of Ramsey, Schur, and Van der Waerden all illustrate the same underlying principle. However, these theorems concern different mathematical objects. We've looked at graphs, additive equations, and arithmetic progressions. Rado's Theorem now analyzes systems of linear equations.

His famous theorem completely characterizes the homogeneous linear equations that possess this Ramsey-theoretic property. In modern terminology, these equations are said to be *partition regular*. Before stating the theorem, we recall the definition of partition regularity.

Definition 2.11. A linear equation is said to be **partition regular** over $\mathbb{N}$ if every finite coloring of the positive integers contains a monochromatic solution to the system.

With this terminology in place, we may now state Rado's Theorem.

Theorem 2.12 (Rado's Theorem [5])**.** *The homogeneous linear equation*

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$$

is partition regular over $\mathbb{N}$ if and only if there exists a nonempty subset

$$J \subseteq \{1, 2, \ldots, n\}$$

such that

$$\sum_{j \in J} c_j = 0.$$

Remarkably, this simple algebraic criterion completely determines whether a homogeneous linear equation is partition regular over the positive integers. Rather than studying each equation individually, Rado's Theorem reduces the problem to checking whether the coefficients satisfy a straightforward algebraic condition.

One of the strengths of Rado's Theorem is that it unifies many of the classical results discussed earlier. For example, Schur's Theorem concerns the equation

$$x + y - z = 0,$$

whose coefficients are

$$1,\ 1,\ -1.$$

Since

$$1 + (-1) = 0,$$

Rado's Theorem immediately implies that the equation is partition regular, recovering Schur's Theorem as a special case. Many other classical results in arithmetic Ramsey Theory can likewise be viewed as consequences of Rado's characterization.

Although Rado's Theorem completely answers the qualitative question of whether monochromatic solutions must exist, it leaves open an equally natural quantitative question. Can we determine how many integers must be colored before a monochromatic solution is guaranteed?

Answering this question leads to the notion of a *Rado number*, the primary object of study for the remainder of this survey.

2.5 Rado Numbers

Rado's Theorem determines whether a homogeneous linear equation is guaranteed to admit monochromatic solutions under every finite coloring of the positive integers. However, it does not specify when such solutions must first appear. As with Schur numbers and Van der Waerden numbers, one may ask for the smallest finite interval on which monochromatic solutions become unavoidable.

This motivates the following definition.

Definition 2.13. Let

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = 0$$

be a linear equation, and let r be a positive integer. The **r-color Rado number**, denoted

$$R_r(a_1x_1 + a_2x_2 + \cdots + a_mx_m = 0),$$

is the smallest positive integer N such that every r-coloring of

$$[1, 2, \ldots, N]$$

contains a monochromatic solution to the equation $a_1x_1+a_2x_2+\cdots+a_mx_m = 0$.

In other words, a Rado number measures precisely how large a finite interval must be before every r-coloring is forced to contain a monochromatic solution. Just as Schur numbers quantify Schur's Theorem and Van der Waerden numbers quantify Van der Waerden's Theorem, Rado numbers provide the quantitative analogue of Rado's Theorem.

Determining exact Rado numbers has become one of the central problems in arithmetic Ramsey Theory. Although Rado's Theorem completely characterizes the partition regular equations, computing their corresponding Rado numbers is often considerably more difficult. Exact values are known for only certain families of equations, while many natural cases remain open. Over the past several decades, researchers have developed a wide variety of combinatorial and computational techniques to determine new Rado numbers. The remainder of this survey explores these developments, focusing on the 2-color Rado numbers, beginning with classical results and concluding with modern approaches based on SAT solvers.

3 Literature Review

3.1 Two-Color Rado Numbers for $a(x + y) = bz$

After introducing Rado numbers, one of the first natural research directions was to determine their exact values for broad families of linear equations. A significant contribution in this direction was made by Harborth and Maasberg in 1999 [6], who completely determined the two-color Rado numbers for the family of equations

$$a(x + y) = bz,$$

where $a, b \in \mathbb{Z}^+$.

Since multiplying both sides of the equation by a common factor does not change its integer solutions, the authors assume throughout that

$$\gcd(a, b) = 1.$$

This allows every equation in the family to be represented in its simplest form.

The main result of the paper is a complete classification of the two-color Rado numbers

$$R_2(a(x + y) = bz)$$

for every coprime pair (a, b). Rather than producing a single closed formula, the answer depends on the arithmetic relationship between a and b. Several distinct behaviors emerge:

- When $b = 1$ or $b = 2$, the Rado numbers are given by relatively simple closed-form expressions.
- When $b = 3$, the situation becomes considerably more intricate. The exact value depends on the residue class of a modulo 9, resulting in several different formulas.
- For $b \geq 4$, the behavior is determined primarily by the ratio a/b, producing different formulas depending on whether a is smaller than, approximately equal to, or larger than b.

Rather than reproducing every case of the classification, we highlight two representative examples:

$$R_2(x + y = z) = 5,$$

recovering the classical two-color Schur number, and

$$R_2(2(x + y) = z) = 17,$$

illustrating how rapidly these values can increase even after a small change in the coefficients.

The proof follows the standard strategy that appears throughout much of the Rado number literature. For each case, the authors first construct explicit two-colorings of

$$\{1, 2, \ldots, N - 1\}$$

that avoid monochromatic solutions, establishing lower bounds. They then prove matching upper bounds by showing that every two-coloring of

$$\{1, 2, \ldots, N\}$$

necessarily contains a monochromatic solution. Since the upper and lower bounds coincide in every case, the exact Rado numbers are obtained.

This paper represents one of the earliest complete classifications of an infinite family of Rado numbers. More importantly, it established many of the proof techniques that would later be adapted to increasingly complicated equations involving subtraction, additional variables, and non-homogeneous terms.

In the next paper, we examine one such extension, where the authors study the closely related family of equations

$$a(x - y) = bz.$$

3.2 A Note on the Two-Color Rado Numbers for $a(x-y) = bz$

Following the work of Harborth and Maasberg, researchers began investigating other infinite families of linear equations in search of exact Rado numbers. One particularly natural extension is the family

$$a(x-y) = bz,$$

where $a, b \in \mathbb{Z}^+$. Although this equation differs from $a(x+y) = bz$ by only a single sign, its behavior is surprisingly different and requires new combinatorial techniques.

Gasarch, Moriarty, and Tumma [7] initiated the study of this family by establishing several lower bounds for the corresponding two-color Rado numbers. In particular, they proved that whenever

$$1 < a < b,$$

the inequality

$$R_2(a(x-y) = bz) \geq b^2 + b + 1$$

holds, and they conjectured that this lower bound was in fact the exact value.

In 2014, Yao [8] confirmed this conjecture by proving matching upper bounds, thereby obtaining a complete characterization of the two-color Rado numbers for the equation $a(x-y) = bz$. Yao proved the following result (Theorem 1.3 in [8]).

Theorem 3.1 (Yao [8])**.**

$$R_2(a(x-y) = bz) = \begin{cases} a^2, & 1 \leq b < a, \\ b^2 + b + 1, & 1 < a < b. \end{cases}$$

One striking feature of this theorem is its simplicity. Unlike the previous family $a(x+y) = bz$, whose exact Rado numbers require numerous case distinctions, the equation $a(x-y) = bz$ is completely described by only two cases depending on whether $a > b$ or $a < b$. This contrast illustrates how even small changes to a linear equation can dramatically alter the complexity of its associated Rado numbers.

For example, if $a = 2$ and $b = 1$, then

$$R_2(2(x-y) = z) = 4,$$

meaning that every two-coloring of

$$\{1, 2, 3, 4\}$$

contains a monochromatic solution to

$$2(x-y) = z.$$

On the other hand, choosing $a = 2$ and $b = 3$ gives

$$R_2(2(x - y) = 3z) = 13,$$

showing that changing only one coefficient can significantly increase the size of the smallest interval guaranteeing a monochromatic solution.

Yao's theorem represents one of the earliest complete classifications of an infinite family of two-color Rado numbers. More importantly, it shows that minor changes to a linear equation can lead to dramatically different combinatorial behavior. This sensitivity to the coefficients continues to appear throughout the literature and motivates the study of increasingly general families of equations.

3.3 Determination of the Two-Color Rado Number for $a_1x_1 + \cdots + a_mx_m = x_0$

Many of the results discussed thus far focus on specific families of equations whose coefficients satisfy particular arithmetic relationships. A natural question is whether one can obtain a general formula for an entire class of linear equations rather than studying each family separately.

In 2008, Guo and Sun [9] answered this question for the equation

$$a_1x_1 + \cdots + a_mx_m = x_0,$$

where $a_1, \ldots, a_m$ are positive integers and $m \geq 2$. Their work resolved a conjecture of Hopkins and Schaal [10] and provided one of the broadest exact formulas known for two-color Rado numbers.

Before this result, exact formulas had been obtained only for special cases. For example, Beutelspacher and Brestovansky [11] showed that

$$R_2(x_1 + \cdots + x_m = x_0) = m^2 + m - 1,$$

while Abbott [12] later proved that

$$R_2(a(x_1 + \cdots + x_m) = x_0) = a^3m^2 + am - a.$$

These results suggested that the two-color Rado number might depend on the coefficients in a systematic way, motivating the search for a general formula. Guo and Sun proved the following result (Theorem 1.1 in [9]).

Theorem 3.2 (Guo and Sun [9]). *Let $m \geq 2$ and let*

$$a_1, \ldots, a_m \in \mathbb{Z}^+.$$

Define

$$a = \min\{a_1, \ldots, a_m\} \text{ and } b = \textstyle\sum_{i=1}^m a_i - a.$$

Then

$$R_2(a_1x_1 + \cdots + a_mx_m = x_0) = a(a + b)^2 + b.$$

Since

$$a + b = \sum_{i=1}^{m} a_i,$$

the theorem shows that the exact two-color Rado number depends only on the sum of the coefficients together with the smallest coefficient. This is somewhat surprising, as one might expect the individual coefficients to play a more complicated role.

Moreover, the theorem immediately recovers several previously known results by choosing appropriate values of the coefficients. Consequently, it provides a unified framework encompassing many earlier computations as special cases.

This result represents an important milestone in the study of Rado numbers. Rather than determining the Rado number for a particular family of equations, Guo and Sun obtained a closed-form expression valid for every equation of the form

$$a_1 x_1 + \cdots + a_m x_m = x_0,$$

demonstrating that broad classes of Rado numbers can sometimes admit remarkably simple formulas.

3.4 Two-Color Rado Number of $x + y + c = kz$ for Large c

The papers discussed thus far focus exclusively on *homogeneous* linear equations, where every term contains a variable. A natural next step is to investigate *non-homogeneous* equations, obtained by introducing a constant term. One important example is the family

$$x + y + c = kz,$$

where c is a fixed positive integer and $k \geq 2$.

Unlike the homogeneous equations considered previously, the existence of a Rado number is no longer guaranteed. Instead, it depends on both the coefficient k and the parity of the constant term c. In 2023, Kim et al. [13] proved that the equation is two-regular if and only if either k is odd or c is even. This completely characterizes when the corresponding two-color Rado number exists.

Having established the existence criterion, the authors turn to the problem of determining the exact Rado number. Their main result shows that when the constant term is sufficiently large, the answer is given by a remarkably concise closed-form formula. Kim et al. proved the following result (Theorem 1 in [13]).

Theorem 3.3 (Kim et al. [13]). *If $k \geq 5$, $c \geq 2k^3 + 2k^2 - k$, and either k is odd or c is even, then*

$$R_2(x + y + c = kz) = \left\lceil \frac{2\left\lceil \frac{c+2}{k} \right\rceil + c}{k} \right\rceil.$$

This theorem illustrates an important shift in the study of Rado numbers. Rather than considering only homogeneous equations, it demonstrates that exact formulas can also be obtained for broad families of non-homogeneous equations. Moreover, the result shows that once the constant term becomes sufficiently large, the behavior of the Rado number follows a predictable pattern that can be described by a single explicit formula.

This work significantly broadened the scope of Rado number research, showing that many of the techniques developed for homogeneous equations can be successfully adapted to the non-homogeneous setting.

3.5 On the Two-Color Rado Number for $x_1 + ax_2 - x_3 = c$

The study of non-homogeneous Rado numbers has expanded considerably in recent years, with researchers investigating increasingly general families of linear equations. Dwivedi and Tripathi made one such contribution in 2020 [14], who studied the family

$$x_1 + ax_2 - x_3 = c,$$

where a is a positive integer and c is any integer.

Their work builds upon earlier results in 2012 of Schaal and Zinter [15], who investigated the special case

$$x_1 + 3x_2 - x_3 = c.$$

For this equation, Schaal and Zinter established sharp lower bounds together with explicit upper bounds depending on the congruence class and parity of the constant term c. In several congruence classes, these bounds coincide, yielding exact two-color Rado numbers.

Motivated by these results, Dwivedi and Tripathi sought to understand what happens when the coefficient of x_2 is allowed to vary. Rather than studying a single equation, they considered the entire family

$$x_1 + ax_2 - x_3 = c,$$

greatly extending the scope of the problem.

The authors obtain necessary and sufficient conditions for the existence of two-color Rado numbers, prove general upper and lower bounds, and derive several closed-form formulas for important classes of parameters. In particular, they determine exact values for sufficiently negative constants when a is odd, for all non-positive multiples of a, and for certain small positive multiples of a.

Unlike many earlier papers that focused on a single equation or a narrowly defined family, this work demonstrates how techniques developed for specific cases can be generalized to much broader classes of non-homogeneous equations. It also illustrates the idea that exact formulas can be obtained in many situations, but a complete characterization for arbitrary values of a and c remains elusive.

4 Introduction to SAT Solvers for Rado Numbers

Many of the exact Rado numbers discussed in the previous section were obtained through intricate combinatorial arguments. As researchers began studying larger equations and more complicated families of colorings, purely theoretical approaches often became increasingly difficult. Consequently, computational methods emerged as an important tool in the study of Rado numbers. Among the most successful of these methods are SAT solvers, which allow coloring problems to be translated into Boolean satisfiability problems. Before discussing their applications to Rado numbers, we briefly review the basic ideas behind SAT solving.

4.1 What is a SAT Solver?

Before explaining how SAT solvers are used to compute Rado numbers, we first introduce several basic concepts from Boolean logic.

Definition 4.1. A **Boolean formula** is a logical expression built from Boolean variables, such as

$$x, y, z, \ldots$$

using the Boolean operations

$$\text{AND}, \text{ OR}, \text{ and NOT}.$$

A Boolean formula evaluates to either `True` or `False`.

Example 4.2. The expression

$$(x \wedge y) \vee (x \wedge \neg y)$$

is a Boolean formula.

A **SAT solver** is a computer program that determines whether a Boolean formula has a satisfying assignment. In other words, it decides whether truth values can be assigned to the variables so that the entire formula evaluates to `True`.

A Boolean formula is said to be *satisfiable* if such an assignment exists; otherwise, it is called *unsatisfiable*.

For example, the Boolean formula above is satisfiable since there exist truth assignments for x and y that make the expression true. If a SAT solver returns `SATISFIABLE`, then a satisfying assignment exists. If it returns `UNSATISFIABLE`, then no assignment of truth values satisfies the formula.

SAT solvers have become powerful tools in combinatorics because many existence problems can be reformulated as questions about whether a suitable Boolean formula is satisfiable. Once such an encoding is constructed, highly optimized SAT solvers can search spaces far too large for direct examination.

In particular, they provide an efficient method for computing and verifying Rado numbers.

Recall that the 2-color Rado number, $R_2(\varepsilon)$, is the smallest positive integer n such that every 2-coloring of $[n]$ contains a monochromatic solution to the equation ε.

To apply a SAT solver, we encode this coloring problem as a Boolean formula. For each integer

$$i \in [n],$$

we introduce Boolean variables representing the color assigned to i. The Boolean formula is then constructed so that every solution to ε contained in $[n]$ is prevented from being monochromatic.

Consequently, a satisfying assignment of the Boolean formula corresponds exactly to a valid coloring of $[n]$ that contains no monochromatic solution to ε.

The overall procedure is as follows.

1. Choose a positive integer n.
2. Construct the Boolean formula corresponding to all colorings of $[n]$ that avoid monochromatic solutions to ε.
3. Run a SAT solver on the resulting formula.
4. If the solver returns `SATISFIABLE`, then there exists a coloring of $[n]$ with no monochromatic solution. Therefore,

$$R_2(\varepsilon) > n.$$

5. If the solver returns `UNSATISFIABLE`, then every coloring of $[n]$ contains a monochromatic solution. Therefore,

$$R_2(\varepsilon) \leq n.$$

By repeatedly testing different values of n, one can determine the smallest integer for which the Boolean formula becomes unsatisfiable. This value is precisely the Rado number. Modern SAT solvers can analyze enormous search spaces far more efficiently than exhaustive enumeration, making them indispensable tools for computing many Rado numbers that would otherwise be infeasible to determine by hand.

4.2 A Simple Example

To illustrate how a SAT solver is used, consider the equation

$$x + y = z,$$

and suppose we wish to determine whether there exists a 2-coloring of

$$\{1, 2, 3, 4\}$$

with no monochromatic solution.

For each integer $i \in \{1, 2, 3, 4\}$, introduce a Boolean variable C_i. We interpret

$$C_i = \texttt{True}$$

to mean that i is colored red, while

$$C_i = \texttt{False}$$

means that i is colored blue.

Next, list every solution to

$$x + y = z$$

contained in $\{1, 2, 3, 4\}$. These are

$$(1, 1, 2), \qquad (1, 2, 3), \qquad (2, 1, 3), \qquad (1, 3, 4), \qquad (3, 1, 4), \qquad (2, 2, 4).$$

For each solution, we add Boolean constraints that prevent all three integers from receiving the same color.

For example, the solution

$$(1, 2, 3)$$

produces the clauses

$$(C_1 \vee C_2 \vee C_3) \text{ and } (\neg C_1 \vee \neg C_2 \vee \neg C_3).$$

The first clause guarantees that at least one of the three numbers is red, while the second guarantees that at least one is blue. Together, these clauses prevent the solution from being monochromatic.

After simplifying repeated literals, the solution

$$(2, 2, 4)$$

yields the clauses

$$(C_2 \vee C_4) \text{ and } (\neg C_2 \vee \neg C_4),$$

which force 2 and 4 to receive different colors.

Repeating this construction for every solution produces a single Boolean formula. A SAT solver then determines whether this formula is satisfiable.

For example, the coloring

$$\{\color{red}{1}, \color{blue}{2}, \color{blue}{3}, \color{red}{4}\}$$

contains no monochromatic solution to

$$x + y = z.$$

Thus the corresponding Boolean formula is satisfiable. Consequently,

$$R_2(x + y = z) > 4.$$

Indeed, Schur's Theorem implies that

$$R_2(x + y = z) = 5.$$

This example illustrates the central idea behind SAT-based approaches to Ramsey theory. Rather than searching directly through all possible colorings, one encodes the coloring problem as a Boolean formula whose satisfying assignments correspond to colorings that avoid monochromatic solutions. Modern SAT solvers can then determine whether such colorings exist, often exploring search spaces that are far too large for exhaustive enumeration. In recent years, these methods have played an increasingly important role in the computation of exact Rado numbers and related Ramsey-theoretic quantities.

4.3 Rado Numbers for $x + y = kz$: Computation, Closed-Form Formula, and Complete Proof

The increasing use of SAT solvers in Ramsey theory has made it possible to investigate families of Rado numbers that would be difficult to analyze using purely combinatorial methods. A recent example is the work of Towell [16], who studied the equation

$$x + y = kz,$$

under the additional requirement that the variables x, y, and z be distinct.

The project began as a computational investigation. Using SAT solvers, Towell computed the corresponding two-color Rado numbers for every value of

$$1 \le k \le 500.$$

The resulting data revealed a striking pattern, suggesting that the Rado numbers might admit a simple closed-form expression depending only on the parity of k. Motivated by these computations, Towell then developed a complete proof establishing the observed formula for all sufficiently large values of k.

The main result is the following (Theorem 1.2 in [16]):

Theorem 4.3 (Towell [16]). *For every $k \ge 8$,*

$$R_2(x + y = kz) = \begin{cases} \frac{k(k+3)}{2}, & \text{if } k \text{ is odd,} \\ \frac{k^2+2k+2}{2}, & \text{if } k \text{ is even.} \end{cases}$$

One particularly interesting aspect of this result is that the behavior of the Rado number depends entirely on the parity of k. Despite the apparent simplicity of the equation, the proof combines extensive computational evidence with rigorous theoretical analysis.

More broadly, this work illustrates a growing trend in modern Ramsey theory. Recently, computation has been increasingly used not merely to verify known results, but also to discover new patterns and formulate conjectures. SAT solvers can rapidly explore enormous coloring spaces, producing data that would be impractical to obtain by hand. These computations then guide the search for rigorous proofs, creating a productive interaction between experimentation and theory.

Viewed in the broader context of this survey, Towell's result serves as a natural culmination of the development of Rado numbers. Beginning with Rado's characterization of partition-regular equations, researchers sought exact formulas for increasingly general families of equations. More recently, computational tools such as SAT solvers have become essential in this pursuit. Towell's work demonstrates how computational methods can lead not only to new numerical data, but also to complete mathematical proofs and infinite families of exact Rado numbers.

5 Conclusion

Since Richard Rado's work in 1933, the study of partition regular linear equations has grown into a vibrant area of arithmetic Ramsey theory. Rado's Theorem provides a complete characterization of when a homogeneous linear equation must admit monochromatic solutions under every finite coloring of the positive integers. However, determining the corresponding Rado numbers remains a far more difficult problem. Despite significant progress over the past century, exact values are known only for certain classes of equations, and many natural cases remain open.

In this survey, we traced the historical development of the subject through the foundational contributions of Ramsey, Schur, and Van der Waerden before presenting Rado's characterization of partition regular equations. We then examined a variety of results concerning exact two-color Rado numbers, including both homogeneous and non-homogeneous equations. These results illustrate the diverse techniques that have been developed to study Rado numbers, ranging from intricate combinatorial arguments and explicit coloring constructions to general closed-form formulas for infinite families of equations.

More recently, computational methods have become an increasingly important part of the subject. As demonstrated by the use of SAT solvers, modern computation can efficiently explore enormous coloring spaces, uncover patterns, generate conjectures, and even guide the development of rigorous mathematical proofs. The growing interaction between theoretical and computational approaches has expanded the range of problems that can be investigated and has opened new avenues for research.

Many questions remain unanswered. Determining additional exact Rado numbers, extending results to colorings with more than two colors, studying larger systems of linear equations, and developing new computational techniques all represent active areas of investigation. As both theoretical and computational tools continue to advance, the study of Rado numbers remains a rich source of challenging problems and a compelling example of the interplay between combinatorics, number theory, and computer-assisted mathematics.

6 Acknowledgements

A huge thank you to the NSF for sponsoring REU-CAAR and this work under Grant No. 2447466. I want to thank my mentor, William Gasarch, for his mentorship throughout the program. I would also like to thank my REU group members, Hridhaan Banerjee, Ryan Hoff, Eva Wu, Gary Hu, Sandy Lin, Yumo Liu, and Tane Park for their support and guidance. Last but certainly not least, I want to thank my mentors at the University of Wisconsin-La Crosse, Chad Vidden and Nathan Warnberg, for helping me get to where I am now and for their continued support.

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