General Upper and Lower Bounds for Some Rado Numbers

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September 1, 2026

Abstract

Here, we provide an algorithm for getting upper bounds on 2-color Rado Numbers for a homogeneous linear equation and a formula for a lower bound for c-color Rado numbers for a homogeneous system of linear equations. Additionally, we provide explicit examples of these results being applied to find bounds on Rado Numbers. We wish to not only provide the algorithm and formula, but also the reasoning behind their constructions.

1 Definitions and Notation

Note: Here, $0 \notin \mathbb{N}$. In other words, $\mathbb{N} = \mathbb{Z}^+$.

Definition 1.1. $[a..b] := \{m : (m \in \mathbb{Z}) \wedge (a \leq m \leq b)\}$

Definition 1.2. $[n] := [1..n]$. More colloquially, $[n]$ is defined to mean $\{1, 2, \ldots, n\}$

Definition 1.3 (Coloring function χ_c)**.** $\chi_c(n) : [n] \to [c]$. Instead of thinking of the codomain of χ_c as being naturals till c, we often denote them as colors R, B, etc. for ease of understanding.

Definition 1.4 (c-color Rado Number)**.** Consider the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. The **c-color Rado number**, denoted

$$R_c(a_1, a_2, \cdots, a_n),$$

is the smallest positive integer N such that every c-coloring of $[N]$ contains a monochromatic solution to the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. Similarly,

$$R_c(A)$$

is the smallest positive integer N such that every c-coloring of $[N]$ contains a monochromatic solution to the system $A\mathbf{x} = \mathbf{0}$.

2 Naive Approach To Upper Bounds For 2-Color Rado Numbers

We wish to determine a monochromatic solution is guaranteed under a 2-coloring of the naturals in cases where Rado's Theorem does not apply. A naive way to go about this is to check if every coloring the subset of the naturals till some upper bound yields a monochromatic solution. If so, the equation is 2-regular with an obvious upper bound on the 2-color Rado number. This approach has some merit, but depending on our goal, we can do better.

Pros: It can be generalized easily to c colorings, systems of equations, and non-linear equations. Furthermore, it can be intelligently used to get the exact 2-color Rado number

Cons: It doesn't necessarily give good bounds and has horrible time complexity. Even while employing intelligent pruning, this issue still holds.

Below, we describe a better, albeit less general, approach.

3 Example and Motivation for Upper Bound Algorithm

Consider the equation $x_1+2x_2-4x_3$. We wish to play a game where we try to avoid a monochromatic solution under a 2-coloring of $\mathbb{N}$.

Assume that $\chi_2(4) = R$ WLOG.

$$4 + 2(4) - 4(3) = 0 \implies \chi_2(3) = B \tag{1}$$
$$4 + 2(2) - 4(2) = 0 \implies \chi_2(2) = B \tag{2}$$

$$2 + 2(3) - 4(2) = 0$$

Therefore, we find that $R_2(1, 2, -4) \leq 4$. In fact, it turns out that $R_2(1, 2, -4) = 4$.

We wish to turn this sort of thing into an algorithm that deduces upper bounds for 2-color Rado Numbers. Note that Rado's Second Theorem states that all linear homogeneous equations with integer coefficients that are not all the same sign and that have at least three variables are 2-regular (meaning that all 2-colorings of $\mathbb{N}$ contain a monochromatic solution to the equation) [3][2].

4 Algorithm in Detail

Below we have a high level description of the algorithm:

1. Assume that $\exists b \in \mathbb{N}$ s.t. $\chi_2(b) = R$ WLOG.
2. Choose a subset of the $x_i's$ to set to numbers that are known to be the same color
$$\chi_2(x_{j_1}) = \chi_2(x_{j_2}) = \cdots = \chi_2(x_{j_m}),\ m < n$$
3. Set all the remaining $x_i's$ to be the same unknown number y
$$y := x_{k_1} = x_{k_2} = \cdots = x_{k_{n-m}}$$
$$J_x = \{x_{j_1}, x_{j_2}, \dots\},\ K_x = \{x_{k_1}, x_{k_2}, \dots\}$$
$$J_x \cap K_x = \emptyset,\ J_x \cup K_x = \{x_i : 1 \leq i \leq n\}$$
4. Solve for y and set it to the opposite color
$$J = \{j_1, j_2, \dots, j_m\}$$
$$y = -\frac{\sum_{j\in J} a_j x_j}{\sum_{k\in([n]-J)} a_k}$$
$$\chi_2(y) = \begin{cases} R & \text{if } \chi_2(x_j) = B \\ B & \text{if } \chi_2(x_j) = R \end{cases}$$
5. Repeat steps 2-4 until both $\chi_2(y) = R$ and $\chi_2(y) = B$ are forced for some y
6. Divide the largest item in the deduction chain by the gcd of every number in the deduction chain. The result is an upper bound for the $R_2(a_1, \dots, a_n)$

7. Repeat the process until you have found the deduction chain that results in the least upper bound. Some deduction chains can be pruned partway based on the largest number divided by the gcd of the numbers in the deduction chain for it partway.

That should give intuition as to how it works. The algorithm is detailed below. Note that to avoid confusion, I use $[[n]]$ to denote $\{1, \ldots, n\}$ instead of $[n]$. There are no nested lists in the following algorithm. One-based indexing is used.

procedure UPPERBOUND
  $A \leftarrow [a_1, \ldots, a_n]$
  $RED \leftarrow [b]$ ▷ $b \in \mathbb{N}$ is a starting number, colored R WLOG; RED is the set of naturals colored red so far
  $BLUE \leftarrow []$ ▷ $BLUE$ is the set of naturals colored blue so far
  $MINRED \leftarrow 0$ ▷ Location in RED of last already deduced number
  $MINBLUE \leftarrow 0$ ▷ Location in $BLUE$ of last already deduced number
  $CHAIN \leftarrow \{b \mapsto \{b\}\}$ ▷ $CHAIN$ is a dictionary consisting of the gcd of the deduction chain to reach said number
  $CMAX \leftarrow \{b \mapsto b\}$ ▷ $CHAIN$ is a dictionary consisting of the max of the deduction chain to reach said number
  $MAX_{R_2} \leftarrow \infty$ ▷ Maximum possible value of 2-color Rado number
  while $LENGTH(RED) > MINRED$ **or** $LENGTH(BLUE) > MINBLUE$ **do**
    if $LENGTH(RED) > MINRED$ **then**
      $NEXTLENR \leftarrow LENGTH(RED)$
      for $SOL \in [[0..LENGTH(RED)]]^n \setminus [[0..MINRED]]^n \setminus [[1..LENGTH(RED)]]^n$ **do**
        $TRIED \leftarrow TRIED \cup \{SOL\}$
        $NEWNUM = -\frac{SUM(\{A[IND] \cdot RED[SOL[IND]] : IND \in [[n]] \wedge SOL[IND] \neq 0\})}{SUM(\{A[IND] : IND \in [[n]] \wedge SOL[IND] = 0\})}$
        $TEMP \leftarrow \max\left(\left(\bigcup_{i \in SOL, i \neq 0} \{CMAX[RED[i]]\}\right) \cup \{NEWNUM\}\right)$
        if $NEWNUM \in CMAX$ **then**
          $CMAX[NEWNUM] \leftarrow \min(CMAX[NEWNUM], TEMP)$
        else
          $CMAX[NEWNUM] \leftarrow TEMP$
        end if
        $CHAIN[NEWNUM] \leftarrow \gcd\left(\left(\bigcup_{i \in SOL, i \neq 0} CHAIN[RED[i]]\right) \cup \{NEWNUM\}\right)$
        if $CMAX[NEWNUM]/CHAIN[NEWNUM] < MAX_{R_2}$ **then**
          if $NEWNUM \in \mathbb{N}$ **then**
            if $NEWNUM \in RED$ **then**
              $MAX_{R_2} \leftarrow CMAX[NEWNUM]/CHAIN[NEWNUM]$
            else
              $BLUE.APPEND(NEWNUM)$
            end if
          end if
        end if
      end for
      $MINRED \leftarrow NEXTLENR$
    end if

    ▷ Below is the same procedure but with blue instead of red
    if $LENGTH(BLUE) > MINBLUE$ **then**

$NEXTLENB \leftarrow LENGTH(BLUE)$
for $SOL \in [[0..LENGTH(RED)]]^n \setminus [[0..MINBLUE]]^n \setminus [[1..LENGTH(BLUE)]]^n$
do
$TRIED \leftarrow TRIED \cup \{SOL\}$
$NEWNUM = -\frac{SUM(\{A[IND] \cdot BLUE[SOL[IND]] : IND \in [[n]] \wedge SOL[IND] \neq 0\})}{SUM(\{A[IND] : IND \in [[n]] \wedge SOL[IND] = 0\})}$
$TEMP \leftarrow \max\left(\left(\bigcup_{i \in SOL, i \neq 0} \{CMAX[BLUE[i]]\}\right) \cup \{NEWNUM\}\right)$
if $NEWNUM \in CMAX$ **then**
$CMAX[NEWNUM] \leftarrow \min(CMAX[NEWNUM], TEMP)$
else
$CMAX[NEWNUM] \leftarrow TEMP$
end if
$CHAIN[NEWNUM] \leftarrow \gcd\left(\left(\bigcup_{i \in SOL, i \neq 0} CHAIN[BLUE[i]]\right) \cup \{NEWNUM\}\right)$
if $CMAX[NEWNUM]/CHAIN[NEWNUM] < MAX_{R_2}$ **then**
if $NEWNUM \in \mathbb{N}$ **then**
if $NEWNUM \in BLUE$ **then**
$MAX_{R_2} \leftarrow CMAX[NEWNUM]/CHAIN[NEWNUM]$
else
$RED.APPEND(NEWNUM)$
end if
end if
end if
end for
$MINBLUE \leftarrow NEXTLENB$
end if
end while
return MAX_{R_2}
end procedure

The starting number b should be chosen to maximize the branching that the algorithm undergoes. I recommend something of the form

$$b = \left(\prod_{i \in 2^{[n]} \setminus \emptyset \setminus [n]} \sum_{j \in i} a_j \right)^p$$

where $p \in \mathbb{N}$. For many simple equations, $p = 1$ suffices. The reason this makes a good starting value is because it is divisible by all of the subset sums of the coefficients, meaning values can immediately be deduced. Note that the algorithm has no guarantee of stopping. Practical implementations will likely need to implement a maximum search limit.

5 Outcomes And Expanding The Scope Of The Algorithm

If we were to expand the algorithm to taking into account colorings with a larger number of colors, one would expect that we would just need to keep track of a set of 'candidate colors' for each natural instead of simply assigning them one. This would make it more difficult to encounter contradictions, which reflects the intuition of having a greater difficulty in guaranteeing a monochromatic solution when more colors are used. However, it turns out that this approach is flawed, as deduced colors cannot be used for further deductions because candidate sets disallow monochromaticity analysis.

Instead, if we were to attempt to expand the algorithm to include a consistent system of linear equations instead of just one, we would no longer be able to deduce even 'candidate colors' of naturals, meaning the entire algorithm would break down. The reason for this is that we could assign a certain subset of the variables to one unknown number and solve for it to force colors in the one equation case, but this was only possible because we only had one equation, meaning we were guaranteed a consistent unknown value. When considering a system, we would need to determine as many new variables at once as the rank of the coefficient matrix for the system, which means when the rank is greater than 1, we lose the ability to deduce information about the monochromaticity of the numbers being tested in general.

Any algorithm aimed at tackling the issue of determining whether a non-Rado system of linear equations without constant terms and all non-zero integer coefficients is r-regular would like to check if there exist solutions to the system where all variables are naturals beforehand. This is equivalent to asking whether there exists a positive vector in the null space of the coefficient matrix for the system. There are linear programming methods for determining this.

In short, these generalizations of the algorithm, if possible, would likely require large changes.

6 Lower Bound for c-Color Rado Number for a homogeneous linear system

Theorem 1. A lower bound for the c-color Rado Number for the equation $\sum_{i=1}^{n} a_i x_i = 0$ is

$$\left(\frac{\max\left(\{\gcd\left(\{a_j : 1 \leq j \leq n \wedge j \neq i\}\right) : 1 \leq i \leq n\}\right)}{\gcd\left(\{a_k : 1 \leq k \leq n\}\right)}\right)^c$$

where $a_i \in \mathbb{Z} \setminus \{0\}$.

Proof. Define $q := \gcd(|a_2|, \ldots, |a_n|)$. We rewrite the equation as

$$a_1 x_1 = -q\left(\sum_{i=2}^{n} \frac{a_i}{q} x_i\right)$$

which we further rewrite as

$$\frac{a_1}{\gcd(a_1, q)} x_1 = -\frac{q}{\gcd(a_1, q)}\left(\sum_{i=2}^{n} \frac{a_i}{q} x_i\right).$$

Since $\gcd\left(\frac{a_1}{\gcd(a_1,q)}, \frac{q}{\gcd(a_1,q)}\right) = 1$, taking both sides of the equation mod $\frac{q}{\gcd(a_1,q)}$ gives

$$\frac{a_1}{\gcd(a_1, q)} x_1 \equiv 0 \pmod{\frac{q}{\gcd(a_1, q)}}.$$

From this, we can conclude that $\frac{q}{\gcd(a_1,q)} \mid x_1$. We define the following c-coloring:

$$\chi_c(x) = \begin{cases} d & \text{if } \left(\frac{q}{\gcd(a_1,q)}\right)^d \mid x \text{ and } \left(\frac{q}{\gcd(a_1,q)}\right)^{d+1} \nmid x \text{ for } d \in [0..c-2] \\ c-1 & \text{if } \left(\frac{q}{\gcd(a_1,q)}\right)^{c-1} \mid x \end{cases}$$

Now, we make three observations.

1. $\forall i \in [n], 1 \mid x_i$ (Note that $x_i \in \mathbb{N}$ so $x_i \neq 0$)

2. If there exists some nonnegative integer d such that $\forall x_i \in [2..n], \left(\frac{q}{\gcd(a_1,q)}\right)^d \mid x_i$, then $\left(\frac{q}{\gcd(a_1,q)}\right)^{d+1} \mid x_1$

3. If $\left(\frac{q}{\gcd(a_1,q)}\right)^d \mid x_1$ where $d < c$, then to have a monochromatic solution, we must have $\forall i \in [n], \left(\frac{q}{\gcd(a_1,q)}\right)^d \mid x_i$

By bounded induction, we get that $\left(\frac{q}{\gcd(q,a_1)}\right)^c \mid x_1$, which gives us $R_c(a_1, \ldots, a_n) \geq \left(\frac{q}{\gcd(q,a_1)}\right)^c$. Noting that we can chose which coefficient is a_1 WLOG yields the lower bound of

$$\left(\frac{\max\left(\{\gcd\left(\{a_j : 1 \leq j \leq n \wedge j \neq i\}\right) : 1 \leq i \leq n\}\right)}{\gcd\left(\{a_k : 1 \leq k \leq n\}\right)}\right)^c$$

□

Theorem 2. A lower bound for the c-color Rado number for the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \ldots & a_{m,n} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is

$$\operatorname{lcm}\left(\left\{\frac{\max\left(\{\gcd(\{a_{j,i} : 1 \leq i \leq n \wedge i \neq k\} \setminus \{0\}) : 1 \leq k \leq n\}\right)}{\gcd(\{a_{j,i} : 1 \leq i \leq n\} \setminus \{0\})} : 1 \leq j \leq m\right\}\right)$$

Proof. We will use the same proof sketch as **Theorem 1**. Define $q_j := \gcd(\{|a_{j,2}|, \ldots, |a_{j,n}|\} \setminus \{0\})$. Assuming that $a_{j,1} \neq 0$, we can rewrite the jth equation in the system as

$$a_{j,1}x_1 = -q_j \left(\sum_{i=2}^{n} \frac{a_{j,i}}{q_j} x_{j,i}\right)$$

which we further rewrite as

$$\frac{a_{j,1}}{\gcd(a_{j,1}, q_j)} x_1 = -\frac{q_j}{\gcd(a_{j,1}, q_j)} \left(\sum_{i=2}^{n} \frac{a_{j,i}}{q_j} x_i\right).$$

This means $\forall j \in [m], \frac{q_j}{\gcd(a_{j,1},q_j)} \mid x_1$. Let $r := \operatorname{lcm}\left(\left\{\frac{q_j}{\gcd(a_{j,1},q_j)} : 1 \leq j \leq m\right\}\right)$.

We define the following c-coloring:

$$\chi_c(x) = \begin{cases} d & \text{if } r^d \mid x \text{ and } r^{d+1} \nmid x \text{ for } d \in [0..c-2] \\ c-1 & \text{if } r^{c-1} \mid x \end{cases}$$

Now, we make three observations.

1. $\forall i \in [n], 1 \mid x_i$

2. If there exists some nonnegative integer d such that $\forall x_i \in [2..n], r^d \mid x_i$, then $r^{d+1} \mid x_1$

3. If $r^d \mid x_1$ where $d < c$, then to have a monochromatic solution, we must have $\forall i \in [n], r^d \mid x_i$

By bounded induction, we find that $r^c \mid x_1$, which gives us that $R_c(A) \geq r^c$. Noting that we can choose which variable is x_1 WLOG yields the lower bound of

$$\operatorname{lcm}\left(\left\{\frac{\max\left(\{\gcd(\{a_{j,i} : 1 \leq i \leq n \wedge i \neq k\} \setminus \{0\}) : 1 \leq k \leq n\}\right)}{\gcd(\{a_{j,i} : 1 \leq i \leq n\} \setminus \{0\})} : 1 \leq j \leq m\right\}\right)$$

. ∎

7 Results

In this section is simply a list of results that can be derived from the aforementioned more general results in order to illustrate the utility of them.

Corollary 1. $R_2(2, 3, -6) \leq 9$

Proof. Assume that $\chi_2(2) = R$ WLOG.

$$2(3) + 3(2) - 6(2) = 0 \implies \chi_2(3) = B \tag{3}$$
$$2(3) + 3(4) - 6(3) = 0 \implies \chi_2(4) = R \tag{4}$$
$$2(9) + 3(2) - 6(4) = 0 \implies \chi_2(9) = B \tag{5}$$
$$2(6) + 3(4) - 6(4) = 0 \implies \chi_2(6) = B \tag{6}$$

$$2(9) + 3(6) - 6(6) = 0$$

Therefore, we find that $R_2(2, 3, -6) \leq 9$. ∎

Corollary 2. $R_2(2, 3, -12) \leq 78$

Proof. Assume that $\chi_2(8) = R$ WLOG.

$$2(36) + 3(8) - 12(8) = 0 \implies \chi_2(36) = B \tag{7}$$
$$2(36) + 3(36) - 12(15) = 0 \implies \chi_2(15) = R \tag{8}$$
$$2(78) + 3(8) - 12(15) = 0 \implies \chi_2(78) = B \tag{9}$$
$$2(15) + 3(22) - 12(8) = 0 \implies \chi_2(22) = B \tag{10}$$

$$2(78) + 3(36) - 12(22) = 0$$

Therefore, we find that $R_2(2, 3, -12) \leq 78$. ∎

Corollary 3. $R_2(5, 13, -17, 28) \leq 331$

Proof. Assume that $\chi_2(12) = R$ WLOG.

$$5(12) + 13(99) - 17(99) + 28(12) = 0 \implies \chi_2(99) = B \tag{11}$$
$$5(41) + 13(12) - 17(41) + 28(12) = 0 \implies \chi_2(41) = B \tag{12}$$
$$5(99) + 13(41) - 17(128) + 28(41) = 0 \implies \chi_2(128) = R \tag{13}$$
$$5(128) + 13(244) - 17(244) + 28(12) = 0 \implies \chi_2(244) = B \tag{14}$$
$$5(12) + 13(12) - 17(128) + 28(70) = 0 \implies \chi_2(70) = B \tag{15}$$
$$5(99) + 13(244) - 17(331) + 28(70) = 0 \implies \chi_2(331) = R \tag{16}$$
$$5(70) + 13(41) - 17(215) + 28(99) = 0 \implies \chi_2(215) = R \tag{17}$$

$$5(331) + 13(128) - 17(215) + 28(12) = 0$$

Therefore, we find that $R_2(5, 13, -17, 28) \leq 331$. $\square$

Corollary 4. $R_2(6, 7, -41) \leq 1107$

Proof. Assume that $\chi_2(39) = R$ WLOG.

$$6(221) + 7(39) - 41(39) = 0 \implies \chi_2(221) = B \tag{18}$$
$$6(697) + 7(697) - 41(221) = 0 \implies \chi_2(697) = R \tag{19}$$
$$6(123) + 7(123) - 41(39) = 0 \implies \chi_2(123) = B \tag{20}$$
$$6(39) + 7(195) - 41(39) = 0 \implies \chi_2(195) = B \tag{21}$$
$$6(615) + 7(615) - 41(195) = 0 \implies \chi_2(615) = R \tag{22}$$
$$6(697) + 7(615) - 41(207) = 0 \implies \chi_2(207) = B \tag{23}$$
$$6(123) + 7(1107) - 41(207) = 0 \implies \chi_2(1107) = R \tag{24}$$
$$6(697) + 7(1107) - 41(291) = 0 \implies \chi_2(291) = B \tag{25}$$
$$6(1075) + 7(291) - 41(207) = 0 \implies \chi_2(1075) = R \tag{26}$$
$$6(215) + 7(1075) - 41(215) = 0 \implies \chi_2(215) = B \tag{27}$$
$$6(207) + 7(543) - 41(123) = 0 \implies \chi_2(543) = R \tag{28}$$
$$6(1107) + 7(615) - 41(267) = 0 \implies \chi_2(267) = B \tag{29}$$
$$6(1107) + 7(1107) - 41(351) = 0 \implies \chi_2(351) = B \tag{30}$$
$$6(267) + 7(351) - 41(99) = 0 \implies \chi_2(99) = R \tag{31}$$
$$6(43) + 7(543) - 41(99) = 0 \implies \chi_2(43) = B \tag{32}$$

$$6(43) + 7(215) - 41(43) = 0$$

Therefore, we find that $R_2(6, 7, -41) \leq 1107$. $\square$

Corollary 5. $R_2(6, -7, 16, -17, 26, -27) \leq 6$

Proof. Assume that $\chi_2(2) = R$ WLOG.

$$6(2) - 7(5) + 16(2) - 17(5) + 26(5) - 27(2) = 0 \implies \chi_2(5) = B \tag{33}$$
$$6(5) - 7(5) + 16(6) - 17(5) + 26(6) - 27(6) = 0 \implies \chi_2(6) = R \tag{34}$$
$$6(3) - 7(2) + 16(2) - 17(2) + 26(2) - 27(2) = 0 \implies \chi_2(3) = B \tag{35}$$
$$6(5) - 7(3) + 16(5) - 17(5) + 26(4) - 27(4) = 0 \implies \chi_2(4) = R \tag{36}$$

$$6(6) - 7(4) + 16(4) - 17(4) + 26(4) - 27(4) = 0$$

Therefore, we find that $R_2(6, -7, 16, -17, 26, -27) \leq 6$. $\square$

Corollary 6. $R_2(5, 78, -156) \leq 12168$

Proof. Assume that $\chi_2(50) = R$ WLOG.

$$5(780) + 78(50) - 156(50) = 0 \implies \chi_2(780) = B \tag{37}$$
$$5(12168) + 78(780) - 156(780) = 0 \implies \chi_2(12168) = R \tag{38}$$
$$5(780) + 78(780) - 156(415) = 0 \implies \chi_2(415) = R \tag{39}$$

$$5(12168) + 78(50) - 156(415) = 0$$

Therefore, we find that $R_2(5, 78, -156) \leq 12168$. $\square$

Corollary 7. $R_2(1, a, -a^2) \le a^3 - a^2$ *for* $a \in \mathbb{N}$ *and* $a \ge 2$

Proof. Assume that $\chi_2(a^2-1) = R$ WLOG.

$$\left(a^3-a^2\right) + a\left(a^3-a^2\right) - a^2\left(a^2-1\right) = 0 \implies \chi_2\left(a^3-a^2\right) = B \tag{40}$$
$$a + a\left(a^2-1\right) - a^2\left(a\right) = 0 \implies \chi_2\left(a\right) = B \tag{41}$$

$$\left(a^3-a^2\right) + a\left(a\right) - a^2\left(a\right) = 0$$

Therefore, we find that $R_2(1, a, -a^2) \le a^3 - a^2$ for every integer $a \ge 2$. □

Corollary 8. $R_2(1, a, -a^3) \le a^6 + a^5 - a^4 - 2a^3 + a$ *for* $a \in \mathbb{N}$ *and* $a \ge 2$

Proof. Assume that $\chi_2(a^2+a) = R$ WLOG.

$$\left(a^5+a^4-a^3-a^2\right) + a\left(a^2+a\right) - a^3\left(a^2+a\right) = 0 \implies \chi_2\left(a^5+a^4-a^3-a^2\right) = B \tag{42}$$
$$\left(a^2+a\right) + a\left(a^4+a^3-a-1\right) - a^3\left(a^2+a\right) = 0 \implies \chi_2\left(a^4+a^3-a-1\right) = B \tag{43}$$
$$\left(a^5+a^4-a^3-a^2\right) + a\left(a^6+a^5-a^4-2a^3+a\right) - a^3\left(a^4+a^3-a-1\right) = 0 \implies \tag{44}$$
$$\chi_2\left(a^6+a^5-a^4-2a^3+a\right) = R \tag{45}$$
$$\left(a^4\right) + a\left(a^4\right) - a^3\left(a^2+a\right) = 0 \implies \chi_2\left(a^4\right) = B \tag{46}$$
$$\left(a^4\right) + a\left(a^5+a^4-a^3-a^2\right) - a^3\left(a^3+a^2-1\right) = 0 \implies \chi_2\left(a^3+a^2-1\right) = R \tag{47}$$

$$\left(a^6+a^5-a^4-2a^3+a\right) + a\left(a^3+a^2-1\right) - a^3\left(a^3+a^2-1\right) = 0$$

Therefore, we find that $R_2(1, a, -a^3) \le a^6 + a^5 - a^4 - 2a^3 + a$ for every integer $a \ge 2$. □

Corollary 9. $R_2(1, a, -a^4) \le a^{10} - 2a^7 - a^6 + a^5 + a^4 + a^3 - a^2$ *for* $a \in \mathbb{N}$ *and* $a \ge 2$

Proof. Assume that $\chi_2(a) = R$ WLOG.

$$\left(a^5-a^2\right) + a\left(a\right) - a^4\left(a\right) = 0 \implies \chi_2\left(a^5-a^2\right) = B \tag{48}$$
$$\left(a\right) + a\left(a^4-1\right) - a^4\left(a\right) = 0 \implies \chi_2\left(a^4-1\right) = B \tag{49}$$
$$\left(a^8-a^6-a^4+a^3\right) + a\left(a^5-a^2\right) - a^4\left(a^4-1\right) = 0 \implies \chi_2\left(a^8-a^6-a^4+a^3\right) = R \tag{50}$$
$$\left(a^5-a^2\right) + a\left(a^7-a^4-a^3+a\right) - a^4\left(a^4-1\right) = 0 \implies \chi_2\left(a^7-a^4-a^3+a\right) = R \tag{51}$$
$$\left(a^8-a^6-a^4+a^3\right) + a\left(a^{10}-2a^7-a^6+a^5+a^4+a^3-a^2\right) - a^4\left(a^7-a^4-a^3+a\right) = 0 \implies \tag{52}$$
$$\chi_2\left(a^{10}-2a^7-a^6+a^5+a^4+a^3-a^2\right) = B \tag{53}$$
$$\left(a^7-a^6+a^5-a^4\right) + a\left(a^7-a^6+a^5-a^4\right) - a^4\left(a^4-1\right) = 0 \implies \chi_2\left(a^7-a^6+a^5-a^4\right) = R \tag{54}$$
$$\left(a^9-2a^6+a^3\right) + a\left(a^5-a^2\right) - a^4\left(a^5-a^2\right) = 0 \implies \chi_2\left(a^9-2a^6+a^3\right) = R \tag{55}$$
$$\left(a^7-a^6+a^5-a^4\right) + a\left(a^9-2a^6+a^3\right) - a^4\left(a^6-a^3-a^2+a\right) = 0 \implies \chi_2\left(a^6-a^3-a^2+a\right) = B \tag{56}$$

$$\left(a^{10}-2a^7-a^6+a^5+a^4+a^3-a^2\right) + a\left(a^6-a^3-a^2+a\right) - a^4\left(a^6-a^3-a^2+a\right) = 0$$

Therefore, we find that $R_2(1, a, -a^4) \le a^{10} - 2a^7 - a^6 + a^5 + a^4 + a^3 - a^2$ for every integer $a \ge 2$. □

Corollary 10. $R_2(k, a, -a) \le a^2$ *for* $a, k \in \mathbb{Z} \setminus \{0\}$ *and* $|a| > |k|$

Proof. If $k < 0$, we can multiply both sides of the equation $kx_1 + ax_2 - ax_3 = 0$ by -1 to get an equivalent equation where $k > 0$. If $a < 0$, we can redefine a to be $-a$ to ensure $a > 0$ without altering the equation in any way. Thus, we can assume WLOG that $a, k \in \mathbb{N}$.
Assume that $\chi_2(a^2 - ka) = R$ WLOG.

$$k\left(a^2 - ka\right) + a\left(a^2 - ka\right) - a\left(a^2 - k^2\right) = 0 \implies \chi_2\left(a^2 - k^2\right) = B \tag{57}$$
$$k\left(a^2\right) + a\left(a^2 - ka\right) - a\left(a^2\right) = 0 \implies \chi_2\left(a^2\right) = B \tag{58}$$
$$k\left(ka\right) + a\left(a^2 - k^2\right) - a\left(a^2\right) = 0 \implies \chi_2\left(ka\right) = R \tag{59}$$
$$k\left(a^2 - ka\right) + a\left(a^2 - 2ka + k^2\right) - a\left(a^2 - ka\right) = 0 \implies \chi_2\left(a^2 - 2ka + k^2\right) = B \tag{60}$$
$$k\left(a^2\right) + a\left(a^2 - 2ka + k^2\right) - a\left(a^2 - ka + k^2\right) = 0 \implies \chi_2\left(a^2 - ka + k^2\right) = R \tag{61}$$

$$k\left(ka\right) + a\left(a^2 - ka\right) - a\left(a^2 - ka + k^2\right) = 0$$

Therefore, we find that $R_2(k, a, -a) \leq a^2$ for $a, k \in \mathbb{Z} \setminus \{0\}$ and $|a| > |k|$. □

Corollary 11. *$R_2(k, a, -a) \leq a^2 + 3ka + k^2$ for $a, k \in \mathbb{Z} \setminus \{0\}$ and $|k| > |a|$*

Proof. If $k < 0$, we can multiply both sides of the equation $kx_1 + ax_2 - ax_3 = 0$ by -1 to get an equivalent equation where $k > 0$. If $a < 0$, we can redefine a to be $-a$ to ensure $a > 0$ without altering the equation in any way. Thus, we can assume WLOG that $a, k \in \mathbb{N}$.
Assume that $\chi_2(ka + a^2) = R$ WLOG.

$$k\left(a^2\right) + a\left(a^2\right) - a\left(ka + a^2\right) = 0 \implies \chi_2\left(a^2\right) = B \tag{62}$$
$$k\left(ka + a^2\right) + a\left(ka + a^2\right) - a\left(a^2 + 2ka + k^2\right) = 0 \implies \chi_2\left(a^2 + 2ka + k^2\right) = B \tag{63}$$
$$k\left(ka + 2a^2\right) + a\left(a^2\right) - a\left(a^2 + 2ka + k^2\right) = 0 \implies \chi_2\left(ka + 2a^2\right) = R \tag{64}$$
$$k\left(a^2\right) + a\left(a^2 + 2ka + k^2\right) - a\left(a^2 + 3ka + k^2\right) = 0 \implies \chi_2\left(a^2 + 3ka + k^2\right) = R \tag{65}$$

$$k\left(ka + 2a^2\right) + a\left(ka + a^2\right) - a\left(a^2 + 3ka + k^2\right) = 0$$

Therefore, we find that $R_2(k, a, -a) \leq a^2 + 3ka + k^2$ for $a, k \in \mathbb{Z} \setminus \{0\}$ and $|k| > |a|$. □

Remark 1. *$R_2(a, a, -a) = 5$ for $a \in \mathbb{R} \setminus \{0\}$*

Proof. This is simply Schur's equation and 5 is simply the 2-color Shur number. This remark is only included for completeness. Although this can be easily shown using the deduction chain for an upper bound and an explicit coloring for a lower bound, it seems pointless to do so. □

Corollary 12. *$R_c(k, a, -a) \geq a^c$ where $a, k \in \mathbb{N}$ and $\gcd(a, k) = 1$*

Proof. By **Theorem 1**, the explicit coloring

$$\chi_c(x) = \begin{cases} d & \text{if } a^d \mid x \text{ and } a^{d+1} \nmid x \text{ for } d \in [0..c-2] \\ c-1 & \text{if } a^{c-1} \mid x \end{cases}$$

gives that $x_1 \geq a^c$ in any monochromatic solution to the equation. □

Corollary 13. *$R_2(k, a, -a) = a^2$ if $|a| > |k|$ and $\gcd(a, k) = 1$.*

Proof. For the rest of the proof, it is assumed that $a, k \in \mathbb{Z} \setminus \{0\}$, By **Corollary 10**, $R_2(k, a, -a) \leq a^2$ when $|a| > |k|$. By **Corollary 12**, $R_2(k, a, -a) \geq a^2$ when $\gcd(a, k) = 1$. Thus, $R_2(k, a, -a) = a^2$ when both $|a| > |k|$ and $\gcd(a, k) = 1$. This result was already shown by Gasarch, Moriarty, and Tumma[1]. □

Corollary 14. $R_2(2, 3, -6) \geq 9$

Proof. By **Theorem 1**, $R_2(2, 3, -6) \geq \left(\frac{\gcd(3,6)}{\gcd(2,3,6)}\right)^2 = 9$. $\square$

Corollary 15. $R_2(2, 3, -6) = 9$

Proof. By **Corollary 1**, $R_2(2, 3, -6) \leq 9$. By **Corollary 14**, $R_2(2, 3, -6) \geq 9$. Thus, $R_2(2, 3, -6) = 9$. $\square$

References

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