POLYNOMIAL VAN DER WAERDEN AND ITS GENERALIZATIONS

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Abstract

Ramsey Theory is a field that often relies on elementary techniques, yet its key results in the field are widely scattered throughout the literature. The goal of this paper is to provide a pedagogical review of the polynomial van der Waerden theorem and to discuss its generalizations.

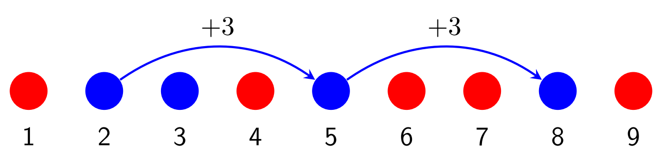
1 Introduction

Let us start with a classical theorem from 1927 due to B. L. van der Waerden [1].

Theorem 1.1 *For all $k, c \in \mathbb{N}$ there exists $W = W(k, c)$ such that for all $COL : [W] \to [c]$, there exist $a, d \in \mathbb{N}$ such that*

$$a, a + d, \ldots, a + (k - 1)d$$

are the same color.



Monochromatic 3-AP (Length 3, Blue)
$a, a + d, a + 2d \to 2, 5, 8$

Figure 1: Illustration of Van der Waerden's Theorem for $k = 3$ and $c = 2$

Some easy cases of this theorem are:

$W(1, c) = 1$. A mono 1-AP is just a single number.
$W(k, 1) = k$. The numbers $1, 2, \ldots, k$ form a mono k-AP.

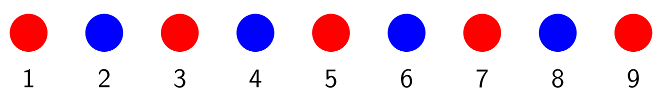
In 1996, Vitaly Bergelson and Alexander Leibman proved a generalization of this result to polynomials [2].

Theorem 1.2 *For all $p_1, \ldots, p_k \in \mathbb{Z}[x]$* ***with*** *$(\forall i)[p_i(0) = 0]$, and $c \in \mathbb{N}$, there exists $W = W(p_1, \ldots, p_k; c)$ such that for all $COL : [W] \to [c]$, there exist $a, d \in \mathbb{N}$ such that*

$$a, a + p_1(d), \ldots, a + p_k(d)$$

are the same color.

This theorem is called the polynomial van der Waerden theorem, and it is often abbreviated PVDW. Note that $p_i(0) = 0$ is crucial. Figure 2 illustrates how the PVDW fails when this condition is dropped.



No two adjacent numbers have the same color
$k = 1$, $p_1(x) = 1$ and $c = 2$

Figure 2: Illustration of why $p_i(0) = 0$ is crucial

We will not prove these results in this paper, since a formal proof of each would require several pages on its own (though we note that elementary proofs of these results exist. A purely combinatorial proof of PVDW was later found by Mark Walters in 2000 [4]). Instead, we focus on a discussion of some of their generalizations.

2 Integral Domains

Def 2.1 An *integral domain* $\mathbb{D} = (D, +, \times)$ consists of a set D and two binary operations $+$, $\times$ such that

1. $+$ and $\times$ are commutative and associative.
2. For all $a, b, c \in D$, $a \times (b + c) = a \times b + a \times c$.
3. There exists $0 \in D$ such that, for all $a \in D$, $a + 0 = a$.
4. There exists $1 \in D$ such that, for all $a \in D$, $a \times 1 = a$.
5. For all $a \in D$ there exists $b \in D$ such that $a + b = 0$.
6. For all $a, b \in D$ such that $a \times b = 0$, $a = 0$ or $b = 0$.

Structures such as $(\mathbb{Z}, +, \times)$, $(\mathbb{Q}, +, \times)$, and $(\mathbb{R}, +, \times)$ are examples of integral domains. A generalized PVDW over such domains was proven in 2005 by Bergelson, Leibman, and McCutcheon [3]:

> **Theorem 2.1** *Let $\mathbb{D}$ be an infinite integral domain. For any $c \in \mathbb{N}$ and any polynomials $p_1(x), \ldots, p_k(x) \in \mathbb{D}[x]$ such that $(\forall i)[p_i(0) = 0]$, there exists a finite subset $\mathbb{D}' \subseteq \mathbb{D}$ such that for any c-coloring $COL : \mathbb{D}' \to [c]$ there exists $a, d \in \mathbb{D}$, $d \neq 0$, such that*
>
> $$a, a + p_1(d), \ldots, a + p_k(d)$$
>
> *are the same color.*

Note that this theorem is immensely powerful: it allows us to generalize the PVDW, originally stated over $\mathbb{Z}$, to continuous objects such as $\mathbb{R}$. This theorem follows as a corollary of the polynomial Hales-Jewett theorem.

3 Intersective Polynomials

The previous section dealt with generalizations to different domains, but let us return to $\mathbb{Z}$. The main question of concern is: how much can we relax the condition $p_i(0) = 0$? Certainly $p_i(0) = 0$ is sufficient for the PVDW to hold, but the proofs mentioned above do not show that it is necessary. In fact, consider the following Proposition.

Proposition 3.1 *For all $p_1, \ldots, p_k \in \mathbb{Z}[x]$ **with** $(\forall i)[p_i(n) = 0]$ **for some fixed integer** n, and $c \in \mathbb{N}$, there exists $W = W(p_1, \ldots, p_k; c)$ such that for all $COL : [W] \to [c]$, there exist $a, d \in \mathbb{N}$ with $d \neq n$ such that*

$$a, a + p_1(d), \ldots, a + p_k(d)$$

are the same color.

Proof: Let $q_i(x) = p_i(x + n)$. The given condition implies $(\forall i)[q_i(0) = 0]$. Before applying the original PVDW on q_i's, we must be careful, since a negative n could force $d \notin \mathbb{N}$. However, since the PVDW also holds for $r_i(x) = q_i(mx)$ for any $m \in \mathbb{N}$, we can force d to be at least m, which resolves this issue, and the Proposition follows from the original PVDW.

This Proposition allows us to shift the zeroes of the polynomials. Now, consider a special family of polynomials.

Proposition 3.2 *Let $p(x) = x^n + 1$ for some $n \in \mathbb{N}$. Then PVDW holds for p and c colors if and only if n is odd, or n is even and $c \in \{1, 2\}$. That is,*

$$PVDW(p; c) = \begin{cases} \textit{true} & \textit{if } n \textit{ odd} \\ \textit{true} & \textit{if } n \textit{ even, } c \in \{1, 2\} \\ \textit{false} & \textit{if } n \textit{ even, } c \geq 3 \end{cases}$$

Proof:

1. This case follows immediately from the Proposition 3.1, since -1 is a root of p.
2. The case $c = 1$ is trivial, and the case $c = 2$ is an easy check using small values of x.
3. It suffices to prove the case $n = 2$ because $x^{2k} + 1 = (x^k)^2 + 1$. Moreover, it suffices to consider $c = 3$, because failure for $c = 3$ automatically implies failure for $c \geq 4$. Consider the following coloring: for an arbitrary integer a, $COL(a) = 1$ iff $a \equiv 1 \pmod 3$, 2 iff $a \equiv 2 \pmod 3$, and 3 iff $a \equiv 0 \pmod 3$.

Under this coloring, $COL(a) = COL(b) \iff a \equiv b \pmod 3$. Therefore,

$$\begin{aligned} COL(a) &= COL(a + x^2 + 1) \\ \implies a &\equiv a + x^2 + 1 \pmod 3 \\ \implies x^2 &\equiv 2 \pmod 3 \end{aligned}$$

This is a contradiction since 2 is not a square mod 3, which finishes the proof.

Although polynomials of the form $x^n + 1$ are a very specific case, this proof shows how powerful the coloring by residue classes can be. In fact, an easy corollary of coloring modulo m is the following.

Proposition 3.3 *Suppose that PVDW holds for some fixed family of polynomials $p_1, \cdots, p_k \in \mathbb{Z}[x]$ for every number of colors $c \in \mathbb{N}$. Then for any $m \in \mathbb{N}$, there exists d such that:*

$$p_1(d) \equiv p_2(d) \equiv \cdots \equiv p_k(d) \equiv 0 \pmod m.$$

Proof: Let m be an arbitrary natural number, and consider coloring by residue mod m, where $COL(a) = COL(b) \iff a \equiv b \pmod m$. Since $a, a + p_1(d), \cdots, a + p_k(d)$ have the same color for some a and d, canceling a gives

$$p_1(d) \equiv p_2(d) \equiv \cdots \equiv p_k(d) \equiv 0 \pmod m.$$

A family of polynomials satisfying the conclusion of Proposition 3.3 is said to be *intersective* (Modding out by 1 does not provide a real constraint). As seen in Proposition 3.2, holding for small values of c does not imply intersectivity. However, it turns out that intersectivity is a necessary and sufficient condition for the PVDW to hold for $p_1, \ldots, p_k$ for every value of $c \in \mathbb{N}$. For all polynomials $p_1, \ldots, p_k \in \mathbb{Z}[x]$, the following are equivalent:

Theorem 3.1 *For all polynomials $p_1, \ldots, p_k \in \mathbb{Z}[x]$, the following are equivalent:*

(i) $p_1, \ldots, p_k$ are intersective.

(ii) for every $c \in \mathbb{N}$ there exists $W = W(p_1, \ldots, p_k; c)$ such that for all $COL : [W] \to [c]$, there exist $a, d \in \mathbb{N}$ such that

$$a, a + p_1(d), \ldots, a + p_k(d)$$

are the same color.

The "(ii) $\Rightarrow$ (i)" direction is exactly Proposition 3.3, and is elementary. The converse was proven by Bergelson, Leibman, and Lesigne in 2008 [5] using techniques from ergodic theory. To date, essentially every known proof of this result seems to require non-pedagogical techniques, such as Fourier-analytic methods or ergodic theory.

4 Conclusion

In this paper we examined the polynomial van der Waerden theorem and two of its generalizations. First, we saw the generalization of PVDW to integral domains, assuming $p_i(0) = 0$. Second, we saw within $\mathbb{Z}$, the condition $p_i(0) = 0$ may be replaced by a weaker condition of intersectivity. A natural follow-up question is whether these two results can be combined — that is, whether the PVDW can be shown to hold over general integral domains under an intersectivity-type condition, with the modding by m replaced by an ideal of the domain. This, however, is currently beyond the reach of my knowledge on the topic.

References

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[5] V. Bergelson, A. Leibman, and E. Lesigne, *Intersective polynomials and the polynomial Szemerédi theorem*, Adv. Math. **219** (2008), no. 1, 369–388.