

The Chromatic Number of E^n : Asymptotics

1 Introduction

I WILL DO THIS LATER AFTER I SEE HOW MUCH IS IN EARLIER CHAPTERS.

2 Elementary Upper Bounds on $\chi(E^n)$

We show $\chi(E^n) \leq (\lceil \sqrt{n} \rceil + 1)^n$. Much better upper bounds are known; however, the proof is elementary and the result gives us a reference point.

The intuition for the coloring used is as follows:

- A small enough cube contains no unit-distance pair. We will color small cubes the same color.
- If a cube C is colored RED, and cube D is more than 1 unit from C , then we are allowed to color D RED.

Theorem 2.1 *For every integer $n \geq 1$,*

$$\chi(E^n) \leq (\lceil \sqrt{n} \rceil + 1)^n.$$

Proof: Put $a = 1/\sqrt{n}$ and partition $\mathbb{R}^n$ into the half-open cubes

$$Q_z = a(z + [0, 1)^n), \quad z \in \mathbb{Z}^n.$$

Set $h = \lceil \sqrt{n} \rceil + 1$ and color Q_z by the vector of residues of the coordinates of z modulo h . This uses h^n colors.

Example 2.2

1. Let $n = 2$, $h = \lceil \sqrt{2} \rceil + 1 = 3$, $a = \frac{1}{\sqrt{2}}$. Let $z = (10, 12)$.

$$Q_z = a(z + [0, 1)^2) = \frac{1}{\sqrt{2}}((10, 20) + [0, 1)^2).$$

We first see what square Q_z is but then note which points on the border are actually not in Q_z .

Corners are:

$$\frac{1}{\sqrt{2}}((10, 20) + (0, 0)) = \left(\frac{10}{\sqrt{2}}, \frac{20}{\sqrt{2}}\right)$$

$$\frac{1}{\sqrt{2}}((10, 20) + (0, 1)) = \left(\frac{10}{\sqrt{2}}, \frac{21}{\sqrt{2}}\right) \text{ (not included)}$$

$$\frac{1}{\sqrt{2}}((10, 20) + (1, 0)) = \left(\frac{11}{\sqrt{2}}, \frac{20}{\sqrt{2}}\right) \text{ (not included)}$$

$$\frac{1}{\sqrt{2}}((10, 20) + (1, 1)) = \left(\frac{21}{\sqrt{2}}, \frac{31}{\sqrt{2}}\right) \text{ (not included)}$$

This square is colored $(10 \pmod 3, 20 \pmod 3) = (1, 2)$.

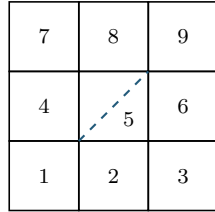
NEAL- DO A PICTURE SIMILAR TO FIGURE 1, LEFT, EXCEPT THAT ITS “INFINITE” AND THE COLORS ARE $(0,0), (0,1), \dots (2,2)$. IN THE CAPTION POINT OUT THAT FOR EACH BOX THE UPPER LINE AND THE RIGHT LINE ARE NOT PART OF IT.

2. Let $n = 3$, $h = \lceil \sqrt{2} \rceil + 1 = 3$, $a = \frac{1}{\sqrt{3}}$. Imagine that you take $\mathbb{E}^3$ and partition it into cubes, similar to how $\mathbb{E}^2$ was partitioned into squares in the last item. Take a cube that is colored (i, j, k) . It has 26 neighbors. Hence we can color $\mathbb{E}^3$ with 27 colors and not get a mono ℓ_2 .

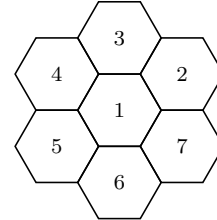
NEAL- DO A PICTURE OF THIS IF YOU CAN.

Two points of the same cube differ by less than a in every coordinate, so their distance is less than $\sqrt{n}a = 1$. If two different cubes have the same color, then their indices differ by at least h in some coordinate. The corresponding coordinates of any two points, one from each cube, therefore differ by strictly more than $(h-1)a \geq 1$. Thus no monochromatic pair is at distance one. ■

Note 2.3 For $n = 2$ the proof uses squares, for $n = 3$ cubes, and so on. The $n = 2$ case of Theorem 2.1 yields $\chi(\mathbb{E}^2) \leq 9$. The hexagonal coloring we presented in Section FILL IN LATER showed $\chi(\mathbb{E}^2) \leq 7$. Can the hexagonal coloring be generalized to get better upper bounds on $\chi(\mathbb{E}^n)$. The answer is a qualified yes. We present results in Section 4.



(a) Squares, 9 colors



(b) Hexagons, 7 colors

Figure 1: Two periodic colorings of the plane. In (a), the cube coloring of Theorem 2.1 at $n = 2$: squares of side $1/\sqrt{2}$, colored by the residues of their index vector modulo $\lceil \sqrt{2} \rceil + 1 = 3$. The dashed diagonal has length one. Half-open boundaries ensure that distances inside one square are strictly smaller than one, while two squares of the same color are three steps apart. In (b), the hexagonal 7-coloring: the hexagons are the Voronoi cells of the triangular lattice, and the seven colors are the cosets of a sublattice of index seven. Note 2.3 discusses what each picture does in higher dimensions.

3 Elementary Lower Bounds on $\chi(\mathbb{E}^n)$

The $n + 1$ points $e_i/\sqrt{2}$ in $\mathbb{R}^{n+1}$ are pairwise at distance one and lie in an affine hyperplane of dimension n . An isometry of that hyperplane with $\mathbb{E}^n$ proves $\chi(\mathbb{E}^n) \geq n + 1$.

A spindle improves the simplex lower bound by one. The construction starts with a configuration that forces two nonadjacent vertices to share a color. Two copies are then positioned so that the forced color equality contradicts a unit-distance edge.

Theorem 3.1 For every integer $n \geq 2$, $\chi(\mathbb{E}^n) \geq n + 2$.

Proof: Take a regular $(n-1)$ -simplex of side one with n vertices in a hyperplane of $\mathbb{E}^n$. Centering the vectors $e_i/\sqrt{2}$ shows that its circumradius squared is $(n-1)/(2n)$. On the perpendicular line through its circumcenter there are two points u, v , one on each side, at distance one from every base vertex. Their separation is

$$D = 2\sqrt{1 - \frac{n-1}{2n}} = \sqrt{\frac{2(n+1)}{n}}, \quad 2D > 1. \quad (1)$$

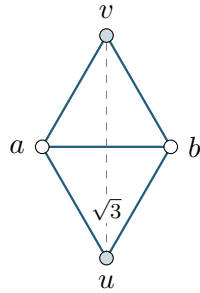
In any proper coloring with $n+1$ colors the n base vertices use distinct colors, so both apices must use the remaining color.

BILL TO NEAL: WHAT IS AN APICE?

Now take two congruent copies of the configuration with their first apices identified at u . Since $n \geq 2$, we may rotate the second copy about u inside a two-dimensional plane through u and v until its other apex v' satisfies $\|v - v'\| = 1$. This is possible because two points on the sphere of radius D about u can have any separation in $[0, 2D]$. Each copy forces its two apices to share a color, so v and v' share the color of u although they are at distance one, a contradiction. ■

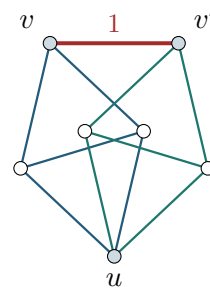
Any further unit distances between the two copies only impose more coloring constraints. The contradiction uses just the edges described in the proof. Figure 2 shows the case $n = 2$, the Moser spindle [3].

(a) A forced equality



Since a, b use two colors, u, v share the third.

(b) The spindle



Both copies force u, v, v' to share a color.

Figure 2: The planar spindle construction. Every solid segment has length one. The dashed segment in (a) has length $\sqrt{3}$ and is not an edge. The two copies in (b) share u and are rotated so that $|v - v'| = 1$. The highlighted edge contradicts the color forced on its endpoints. Crossings without a dot are not vertices.

4 Asymptotic Exponential Upper Bounds on $\chi(\mathbb{E}^n)$

Recall that in Theorem 2.1 we obtained

$$\chi(\mathbb{E}^n) \leq (\lceil \sqrt{n} \rceil + 1)^n$$

using a cube-based coloring. This yields $\chi(\mathbb{E}^2) \leq 9$. However, in Theorem FILL IN LATER we got $\chi(\mathbb{E}^2) \leq 7$ using a hexagonal coloring. Can the hexagonal coloring be generalized to obtain a better upper bound on $\chi(\mathbb{E}^n)$. The answer is roughly yes.

The Eisenstein lattice (defined below) can be viewed as a generalization of the Hexagonal lattice.

Definition 4.1 An Eisenstein lattice is NEAL FILL IN

NEAL- GIVE EXAMPLES AND PICTURES.

Arman, Bondarenko, Prymak, and Radchenko [1] (see after the theorem for some notes on independent discover) used the Eisenstein lattice to prove the following.

Theorem 4.2 1. For all $2 \leq n \leq 38$ and $n \in \{48, 49\}$, $\chi(\mathbb{E}^n) \leq 3^n$.

2. for $n \in \{2, 4, 6, 8, 24\}$, $\chi(\mathbb{E}^n) \leq 7^{n/2}$.

Proof:

■

Note 4.3 NEAL- ARE THE RESULTS OF COULSON AND RADOI-TOTH IND OF ABPR? THAT CAN'T QUITE BE RIGHT IND ABPR WAS IN 2024 AND COULSON, RADOI-TOTH ARE MUCH EARLIER.

Coulson [2], and independently Radoičić and Tóth [5] obtained the following:

1. $\chi(\mathbb{E}^4) \leq 49$.
2. $\chi(\mathbb{E}^5) \leq 140$.
3. For $2 \leq n \leq 38$ and $n \in \{48, 49\}$, $\chi(\mathbb{E}^n) \leq 3^n$.

For some small values of n , the bound of Theorem 4.2 can be improved. That will be the topic of Chapter FILL IN LATER which is on concrete values of $\chi(\mathbb{E}^n)$.

5 Asymptotic Exponential Lower Bounds on $\chi(\mathbb{E}^n)$

References

- [1] A. Arman, A. Bondarenko, A. Prymak, , and D. Radchenko. Upper bounds on chromatic number of $\mathbb{E}^n$ in low dimensions. *Electronic Journal of Combinatorics*, 31(2), 2024.
<https://arxiv.org/abs/2112.13438>.
- [2] D. Coulson. A 15-colouring of 3-space omitting distance one. *Discrete Mathematics*, 256(1-2):83–90, 2002.
- [3] L. Moser and W. Moser. Solution to problem 10. *Canadian Mathematical Bulletin*, 4:187–189, 1961.

- [4] R. Prosanov. A new proof of the larman-rogers upper bound for the chromatic number of the euclidean space. *Discrete Applied Mathematics*, 276:115–120, 2020.
<https://doi.org/10.1016/j.dam.2019.05.020a>.
- [5] R. s Radoiĉić and G. Tóth. Note on the chromatic number of the space. In *Discrete and Computational Geometry: The Goodman-Pollack Festschrift, Algorithms and Combinatorics*, volume 25, pages 695–698. Springer, 2003.