

A Remark on Lower Bounds for the Chromatic Numbers of Spaces of Small Dimension with Metrics ℓ_1 and ℓ_2

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Abstract—A particular class of estimates related to the Nelson–Erdős–Hadwiger problem is studied. For two types of spaces, Euclidean and spaces with metric ℓ_1 , certain series of distance graphs of small dimensions are considered. Independence numbers of such graphs are estimated by using the linear-algebraic method and combinatorial observations. This makes it possible to obtain certain lower bounds for the chromatic numbers of the spaces mentioned above and, for each case, specify a series of graphs leading to the strongest results.

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1. INTRODUCTION AND HISTORY OF THE PROBLEM

The classical Nelson–Erdős–Hadwiger problem, which was posed at the turn of the 1940s and 1950s, consists in finding the quantity $\chi(\mathbb{R}^n)$, called the *chromatic number of the space* $\mathbb{R}^n$ and equal to the least number of colors that can be used to paint all points of $\mathbb{R}^n$ so that the distance between points of the same color is distinct from 1 (see [1]–[7]).

For the quantity $\chi(\mathbb{R}^n)$, a variety of different estimates were obtained for both concrete values of n (see [8]–[22]) and in the asymptotics (see [15], [23], [24]). In particular, the best well-known asymptotic estimates are as follows:

$$(1.239 \dots + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n.$$

As a natural generalization of the problem under consideration, one can consider a similar problem for spaces with other metrics, as well as the case of several forbidden distances between points of the same color (see [25]–[50]). One of the most studied cases involves the Manhattan metric. Let us recall that the metric on $\mathbb{R}^n$ defined by the formula

$$\ell_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \dots + |x_n - y_n|$$

is said to be *Manhattan*. We shall denote the corresponding chromatic number by $\chi(\mathbb{R}^n, \ell_1)$. In this context,

$$\chi(\mathbb{R}^n) = \chi(\mathbb{R}^n, \ell_2).$$

For the metric ℓ_1 , the following asymptotic estimates are known (see [39], [40], [51]):

$$(1.366 \dots + o(1))^n \leq \chi(\mathbb{R}^n) \leq (4 + o(1))^n.$$

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However, for small dimensions, no results have been obtained earlier.

Both asymptotic lower bounds were obtained by using the so-called linear-algebraic method in combinatorics (see [5], [52]). In the present paper, we have succeeded, first, in stating and proving a number of new results based on the linear-algebraic method in their most general form. Second, we have obtained new lower bounds for the chromatic numbers of spaces with the Euclidean and the Manhattan metric of fixed dimension. For the Manhattan metric, this is the first such result.

The paper is organized as follows. In Sec. 2, we discuss well-known results for small dimensions, mentioning some new statements. In Sec. 3, we give the general scheme of proof. Further, in Sec. 4, we state and prove a number of general theorems. In Sec. 5, we bring nontrivial refinements of these theorems for various cases. In Sec. 6, we obtain numerical results for lower bounds for the chromatic numbers of the spaces under consideration. In Sec. 7, we make several remarks related to the results proved in the paper.

2. WELL-KNOWN ESTIMATES FOR SMALL DIMENSIONS AND FORMULATIONS OF NEW RESULTS

As to the chromatic numbers $\chi(\mathbb{R}^n, \ell_1)$ for small dimensions, the authors are unaware of any work dealing with estimates of these quantities. We are the first to propose, for example, the following results:

$$\chi(\mathbb{R}^5, \ell_1) \geq 8, \quad \chi(\mathbb{R}^6, \ell_1) \geq 14, \quad \chi(\mathbb{R}^7, \ell_1) \geq 20, \quad \chi(\mathbb{R}^8, \ell_1) \geq 28, \\ \chi(\mathbb{R}^9, \ell_1) \geq 38, \quad \dots,$$

subject, apparently, to subsequent improvement.

In the case of the metric ℓ_2 , the situation for small dimensions is, on the contrary, well studied. The dimensions $n \leq 12$ have been studied especially thoroughly. In the table given below, we collect all currently known results, with dimensions and references.

Table 1

n	$\chi(\mathbb{R}^n, \ell_2) \geq$	Reference
2	4	[1], [5]
3	6	[17]
4	9	[12]
5	9	[8]
6	12	[11]
7	16	[11]
8	19	[13]
9	21	[14]
10	26	[11], [13]
11	32	[13]
12	36	[11]

In addition, further refinements of certain estimates from the table were recently announced in [18]. Namely,

$$\chi(\mathbb{R}^{10}, \ell_2) \geq 30, \quad \chi(\mathbb{R}^{11}, \ell_2) \geq 35, \quad \chi(\mathbb{R}^{12}, \ell_2) \geq 37.$$

We mention at once that the results for dimensions $n \leq 12$ obtained by our method are weaker than the best known bounds given in Table 1. So, for example, among our best results, there are the estimates $\chi(\mathbb{R}^{11}) \geq 19$ and $\chi(\mathbb{R}^{12}) \geq 22$.

However, the chromatic number has also been studied for $n > 12$. Here the data is rather incomplete. Some results were collected in a table in [2]. In Table 2, we compare those results for $n = 13, \dots, 24$ with ours.

Table 2

n	Estimate in [2]: $\chi(\mathbb{R}^n, \ell_2) \geq$	Our estimate: $\chi(\mathbb{R}^n, \ell_2) \geq$
13	31	26
14	35	31
15	37	37
16	67	46
17	56	56
18	68	68
19	82	82
20	97	97
21	114	114
22	133	139
23	154	171
24	178	209

As we can see, in the present paper, we have obtained the following new estimates:

$$\chi(\mathbb{R}^{22}, \ell_2) \geq 139, \quad \chi(\mathbb{R}^{23}, \ell_2) \geq 171, \quad \chi(\mathbb{R}^{24}, \ell_2) \geq 209.$$

Also, for $n = 16$, an inaccuracy has been removed (obviously, $\chi(\mathbb{R}^{17}) \geq \chi(\mathbb{R}^{16})$).

The results of [2] were obtained by using the so-called $\{0, 1\}$ -vectors (see Sec. 6). In our paper, we also make use of $\{0, 1\}$ -vectors; moreover, we extend our study to cover $\{-1, 0, 1\}$ -vectors and compare the results obtained for small dimensions. This has not been done before.

It is important that, in the paper, we determine the moment (see the table in Sec. 6.4) after which the estimates related to $\{-1, 0, 1\}$ -vectors become the sharpest ones in the Euclidean case. This is what is to be expected, because, in the asymptotic case, it is these vectors that give the best result (see [24]); however, the moment at which $\{-1, 0, 1\}$ -vectors start to dominate as the dimension of the space increases has not been studied up to now.

Many of the quantities $\chi(\mathbb{R}^n, \ell_2)$, $n > 24$, are, apparently, estimated in the present paper for the first time. This has become possible, because the proposed method can be easily scaled to different dimensions.

In Secs. 6.2 and 6.4, results for many small dimensions are collected in tables corresponding to the metrics ℓ_1 and ℓ_2 .

3. THE GENERAL IDEA FOR OBTAINING LOWER BOUNDS FOR THE CHROMATIC NUMBERS OF METRIC SPACES

To obtain lower bounds for the chromatic numbers of the spaces $\mathbb{R}^n$, one often uses the following considerations.

First, consider some graph $G = (V, E)$. By an *independence set* in G we mean a set of vertices from V no two of which are joined by an edge.

By an *independence number of G* we mean the maximum possible cardinality of such a set. The independence number of G is denoted by $\alpha(G)$.

Just as for a space, there is a similar definition of the *chromatic number of a graph* G : this is the minimum possible number of colors that can be used for painting the vertices from V so that no edge from E joins vertices of the same color. This number is denoted by $\chi(G)$.

It is easy to see that, for any coloring excluding edges between same-color vertices of the graph, the sets of same-color vertices are independence sets in the graph G . The minimum number of such “color clusters” in a fixed coloring and the maximum size of clusters are then related, in the terms introduced above, by

$$\chi(G) \geq \frac{|V|}{\alpha(G)}. \quad (3.1)$$

Second, suppose that G is embedded in the space $\mathbb{R}^n$ equipped with its metric, i.e., each vertex in V is identified with some point of the space.

If the vertices of G can be embedded in the space so that the edge set of the graph E coincides with the set of pairs of vertices a given fixed distance apart, then the graph G will be called a *distance graph*. Note that the choice of this distance does not affect the value of the chromatic number: configurations with different “edge lengths” can be taken to one another by a homothety. This allows us to claim that the following relationship between the chromatic number of the whole space and the chromatic number of a finite distance graph embedded in it exists:

$$\chi(\mathbb{R}^n) \geq \chi(G). \quad (3.2)$$

Combining (3.1) with (3.2), we can write

$$\chi(\mathbb{R}^n) \geq \frac{|V|}{\alpha(G)}, \quad (3.3)$$

where it is assumed that $\chi(\mathbb{R}^n) \equiv \chi(\mathbb{R}^n, \rho)$ for the metric ρ of the space, while we are free to choose the graph embedded in the space.

We shall refine these considerations below, in Sec. 6, where a method for choosing distance graphs will be described.

Thus, we have just explained how the problem of finding a lower bound for the chromatic number of a space can be reduced to the problem of finding an upper bound for the independence numbers of finite graphs. In the present paper, the latter problem is studied by using the classical *linear-algebraic method in combinatorics* (see, e.g., [5], [52]); its exposition is given in the next section.

4. THE LINEAR-ALGEBRAIC METHOD

4.1. The Case of the Manhattan Metric ℓ_1

Theorem 1. *Let p be a prime, and let $n \in \mathbb{N}$. Consider an arbitrary set $\Sigma \subset \{-1, 0, 1\}^n$ of vectors such that*

$$\forall \mathbf{x}, \mathbf{y} \in \Sigma \quad \ell_1(\mathbf{x}, \mathbf{y}) \in 2\mathbb{Z}$$

and a subset $\mathcal{F} \subset \Sigma$ such that, for some $\alpha \in \mathbb{N}$ and all $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{F}$,

$$\frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \equiv 0 \pmod{p^\alpha} \iff \mathbf{x} = \mathbf{y},$$

where $n \geq p^\alpha - 1$. Then

$$|\mathcal{F}| \leq \sum_{m=0}^{p^\alpha-1} 2^m \binom{n}{m}.$$

Moreover, if $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, then

$$|\mathcal{F}| \leq \sum_{m=0}^{p^\alpha-1} \binom{n}{m}.$$

Proof. We begin the proof with the following lemma.

Lemma 1. *For all $\mathbf{x}, \mathbf{y} \in \Sigma$, the condition $\ell_1(\mathbf{x}, \mathbf{y})/2 \equiv 0 \pmod{p^\alpha}$ is equivalent to the requirement that $\mathbf{x}$ and $\mathbf{y}$ either coincide or do not lie in $\mathcal{F}$ simultaneously.*

Proof. This requirement is a mere reformulation of the condition. □

For a fixed vector $\mathbf{x} \in \Sigma$, we define the following function of vectors $\mathbf{y} \in \Sigma$:

$$\mathcal{G}_{\mathbf{x}}(\mathbf{y}) = \frac{1}{p^\beta} \prod_{i=1}^{p^\alpha-1} \left(i - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \right),$$

where $\beta \in \mathbb{Z}_{\geq 0}$ is the maximum degree of an occurrence of p in $(p^\alpha - 1)!$, i.e., β is such that $(p^\alpha - 1)!/p^\beta$ is a natural number not divisible by p .

Let us show that, for all $\mathbf{x}, \mathbf{y} \in \Sigma$, we have $\mathcal{G}_{\mathbf{x}}(\mathbf{y}) \in \mathbb{Z}$; moreover, the following statement is valid.

Lemma 2. *The following equivalence holds:*

$$\mathcal{G}_{\mathbf{x}}(\mathbf{y}) \equiv 0 \pmod{p} \quad \Longleftrightarrow \quad \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \not\equiv 0 \pmod{p^\alpha}.$$

Proof. Let $\ell_1(\mathbf{x}, \mathbf{y})/2 \equiv 0 \pmod{p^\alpha}$. Then $\mathcal{G}_{\mathbf{x}}(\mathbf{y}) \not\equiv 0 \pmod{p}$ by construction.

Suppose that, on the contrary, $\ell_1(\mathbf{x}, \mathbf{y})/2 \not\equiv 0 \pmod{p^\alpha}$. We shall verify that, in this case,

$$\mathcal{G}_{\mathbf{x}}(\mathbf{y}) \equiv 0 \pmod{p}.$$

To do this, it suffices to show that

$$p^{\beta+1} \mid \prod_{i=1}^{p^\alpha-1} \left(i - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \right).$$

If $\ell_1(\mathbf{x}, \mathbf{y})/2 \in \{1, \dots, p^\alpha - 1\}$, then the assertion is trivial: the product simply vanishes. We shall assume that $\ell_1(\mathbf{x}, \mathbf{y})/2 > p^\alpha$.

Let us compare the maximum degrees β and $\tilde{\beta}$ of an occurrence of a prime p in $(p^\alpha - 1)!$ and $\prod_{i=1}^{p^\alpha-1} (i - \ell_1(\mathbf{x}, \mathbf{y})/2)$, respectively.

Let $D_Q(S)$ be the number of multiples of $Q \in \mathbb{Z}$ among the elements of a set S of integers. Note that if $L, R, \Delta \in \mathbb{Z}$, $L < R$, and $L - 1 \equiv 0 \pmod{Q}$, then

$$D_Q([L + \Delta, R + \Delta]) \geq D_Q([L, R]),$$

i.e., to minimize the number of multiples of a given number in a closed interval of fixed length with integer coordinates, it suffices to place the left endpoint of the interval on the right of any multiple at a distance of 1 from it.

Using this consideration, we estimate $\tilde{\beta}$:

$$\begin{aligned} \tilde{\beta} &= \sum_{\gamma=1}^{\infty} D_{p^\gamma} \left(\left[1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2}; p^\alpha - 1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \right] \right) \\ &\geq \sum_{\gamma=1}^{\alpha} D_{p^\gamma} \left(\left[1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2}; p^\alpha - 1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \right] \right) \\ &= \sum_{\gamma=1}^{\alpha-1} D_{p^\gamma} \left(\left[1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2}; p^\alpha - 1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \right] \right) + 1 \\ &\geq \sum_{\gamma=1}^{\alpha-1} D_{p^\gamma}([1; p^\alpha - 1]) + 1 = \beta + 1. \end{aligned}$$

In the argument above, we took into account the fact that, for $\ell_1(\mathbf{x}, \mathbf{y})/2 \not\equiv 0 \pmod{p^\alpha}$, the closed interval

$$\left[1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2}; p^\alpha - 1 - \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2}\right]$$

necessarily contains an integer which is a multiple of p^α .

Thus, $\tilde{\beta} > \beta$, which implies the required relation $\mathcal{G}_{\mathbf{x}}(\mathbf{y}) \equiv 0 \pmod{p}$. The lemma is proved. $\square$

Note that, for fixed $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma$, the function

$$\ell_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \dots + |x_n - y_n|$$

can be regarded as an even function of n integer arguments $y_j \in \{-1, 0, 1\}$, which are the coordinates of the vector $\mathbf{y} \in \Sigma$. Here each summand of the form $|x_j - y_j|$, $1 \leq j \leq n$, can be replaced by a polynomial of at most second degree as follows:

$$|x_j - y_j| = \begin{cases} 1 - y_j, & x_j = 1, \\ y_j^2, & x_j = 0, \\ 1 + y_j, & x_j = -1, \end{cases} \quad (4.1)$$

for $y_j \in \{-1, 0, 1\}$.

Thus replacing $\ell_1(\mathbf{x}, \mathbf{y})/2$ by a sum of polynomials q_{x_j} in the variables y_j in the definition of the function $\mathcal{G}_{\mathbf{x}}(\mathbf{y})$, we obtain the following polynomial in $\mathbb{Q}[y_1, \dots, y_n]$:

$$\mathcal{P}_{\mathbf{x}}(\mathbf{y}) = \frac{1}{p^\beta} \prod_{i=1}^{p^\alpha-1} (i - [q_{x_1}(y_1) + \dots + q_{x_n}(y_n)]).$$

Considering an arbitrary argument $\mathbf{y} \in \Sigma$, we note that, by construction, $\mathcal{G}_{\mathbf{x}}(\mathbf{y}) = \mathcal{P}_{\mathbf{x}}(\mathbf{y})$; removing the parentheses from the expression for the polynomial, we see that $\mathcal{P}_{\mathbf{x}}(\mathbf{y})$ can be expressed as a linear combination of monomials each of which contains at most $p^\alpha - 1$ different variables y_j .

At the next step, we use the fact that $y_j^3 = y_j$ for $\mathbf{y} \in \Sigma$. In $\mathcal{P}_{\mathbf{x}}(\mathbf{y})$, replacing each natural degree of an occurrence of the variable $\delta \in \mathbb{Z}_{>0}$ by $2 - (\delta \bmod 2)$, we obtain the new polynomials $\tilde{\mathcal{P}}_{\mathbf{x}}(\mathbf{y})$:

$$\tilde{\mathcal{P}}_{\mathbf{x}}(\mathbf{y}) = \frac{1}{p^\beta} \sum_{m < p^\alpha, \beta_k \leq 2} \gamma_{j_1, \dots, j_m}^{\beta_1, \dots, \beta_m} y_{j_1}^{\beta_1} \dots y_{j_m}^{\beta_m}, \quad (4.2)$$

where $\gamma_{j_1, \dots, j_m}^{\beta_1, \dots, \beta_m} \in \mathbb{Z}$. The monomials of the constructed polynomial contain each variable y_j to the power at most 2; in all, each monomial contains at most $p^\alpha - 1$ different variables. In addition, by construction, the values of the polynomial $\tilde{\mathcal{P}}_{\mathbf{x}}$ and the function $\mathcal{G}_{\mathbf{x}}$ at the vectors $\mathbf{y} \in \Sigma$ coincide. Thus, all the assertions involving the values of $\mathcal{G}_{\mathbf{x}}$ can be carried over in a natural way to the modified polynomials $\tilde{\mathcal{P}}_{\mathbf{x}}$.

Lemma 3. *Let $\mathbf{x}_1, \dots, \mathbf{x}_s \in \Sigma$ be different vectors such that $\ell_1(\mathbf{x}_i, \mathbf{x}_j)/2 \not\equiv 0 \pmod{p^\alpha}$ for all i and j , $i \neq j$. Then the polynomials $\tilde{\mathcal{P}}_{\mathbf{x}_1}, \dots, \tilde{\mathcal{P}}_{\mathbf{x}_s}$ corresponding to these vectors are linearly independent over the field $\mathbb{Q}$.*

Proof. Assume that, on the contrary, for some $c'_1, \dots, c'_s \in \mathbb{Q}$, not all simultaneously zero, we have

$$c'_1 \tilde{\mathcal{P}}_{\mathbf{x}_1} + \dots + c'_s \tilde{\mathcal{P}}_{\mathbf{x}_s} = 0.$$

Then, for all $\mathbf{y} \in \Sigma$, the following equality holds:

$$c'_1 \tilde{\mathcal{P}}_{\mathbf{x}_1}(\mathbf{y}) + \dots + c'_s \tilde{\mathcal{P}}_{\mathbf{x}_s}(\mathbf{y}) = 0.$$

A similar equality also holds for some $c_1, \dots, c_s \in \mathbb{Z}$ such that at least one of these numbers is not divisible by p :

$$c_1 \tilde{\mathcal{P}}_{\mathbf{x}_1}(\mathbf{y}) + \dots + c_s \tilde{\mathcal{P}}_{\mathbf{x}_s}(\mathbf{y}) = 0.$$

Here, to pass from the rational fractions c'_i to the integers c_i , we multiplied the former by the product of their denominators and then divided the resulting numbers by their greatest common divisor.

Substituting $\mathbf{y} = \mathbf{x}_i$, $1 \leq i \leq s$, into this equality, we obtain

$$c_1 \mathcal{G}_{\mathbf{x}_1}(\mathbf{x}_i) + \cdots + c_i \mathcal{G}_{\mathbf{x}_i}(\mathbf{x}_i) + \cdots + c_s \mathcal{G}_{\mathbf{x}_s}(\mathbf{x}_i) = 0.$$

For $j \neq i$, we have $\ell_1(\mathbf{x}_i, \mathbf{x}_j)/2 \not\equiv 0 \pmod{p^\alpha}$, and hence $\mathcal{G}_{\mathbf{x}_j}(\mathbf{x}_i) \equiv 0 \pmod{p}$. Therefore, taking the congruence modulo p , we obtain $c_i \mathcal{G}_{\mathbf{x}_i}(\mathbf{x}_i) \equiv 0 \pmod{p}$, and hence (by construction, $\mathcal{G}_{\mathbf{x}_i}(\mathbf{x}_i) \not\equiv 0 \pmod{p}$) we also have $c_i \equiv 0 \pmod{p}$. Since i is arbitrary, we obtain a contradiction to the choice of c_i , because it was assumed that at least one of these coefficients is not divisible by p . $\square$

Using this lemma, we can estimate the cardinality of the set $\mathcal{F}$ from above by the dimension of some space $\tilde{\mathcal{P}}$ over $\mathbb{Q}$ containing all polynomials $\tilde{\mathcal{P}}_{\mathbf{x}}(\mathbf{y})$ for all $\mathbf{x} \in \Sigma$. We assume that $\tilde{\mathcal{P}}$ is an inclusion-minimal space with this property.

It remains to estimate the dimension of $\tilde{\mathcal{P}}$ from above. To do this, we consider the set consisting of all unique monomials appearing in the sum (4.2). Let us calculate the number of these monomials. We will group all monomials by the number m of unique variables y_{j_k} to a positive power. For fixed m , there exist $\binom{n}{m}$ ways of choosing the positions $j_1, \dots, j_m$; for fixed positions, there are 2^m ways of assigning the powers 1 and 2 to the variables in the monomial. Thus, we obtain

$$|\mathcal{F}| \leq \dim \tilde{\mathcal{P}} = \sum_{m=0}^{p^\alpha-1} 2^m \binom{n}{m},$$

which concludes the proof of the first assertion of the theorem.

It remains to note that, for the case in which $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, the passage from the polynomials $\mathcal{P}$ to the polynomials $\tilde{\mathcal{P}}$ can be performed by virtue of the equality $y_i^2 = y_i$; as a result, the dimension of the space $\tilde{\mathcal{P}}$ can be estimated more precisely, because now all “basis” monomials will contain the variables y_i only to the powers 0 and 1. Thus, we obtain the desired estimate

$$|\mathcal{F}| \leq \dim \tilde{\mathcal{P}} = \sum_{m=0}^{p^\alpha-1} \binom{n}{m},$$

which concludes the proof of the theorem. $\square$

4.2. The Case of the Euclidean Metric ℓ_2

Theorem 2. *Let p be a prime, and let $n \in \mathbb{N}$. Consider an arbitrary set $\Sigma \subset \{-1, 0, 1\}^n$ of vectors such that $(\mathbf{x}, \mathbf{x}) = d$ for all $\mathbf{x} \in \Sigma$ and a subset $\mathcal{F} \subset \Sigma$ such that, for some $\alpha \in \mathbb{N}$ and any different $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{F}$, the following relation holds:*

$$(\mathbf{x}, \mathbf{y}) \not\equiv d \pmod{p^\alpha},$$

where $(\mathbf{x}, \mathbf{y})$ is the Euclidean inner product of vectors in $\mathbb{R}^n$ and $d \in \mathbb{N}$ is a constant. Suppose that $n \geq p^\alpha - 1$.

Then

$$|\mathcal{F}| \leq \sum_{(m_1, m_2) \in \mathcal{A}} \binom{n}{m_1} \binom{n - m_1}{m_2},$$

where

$$\mathcal{A} = \{(m_1, m_2) : m_1, m_2 \in \mathbb{Z}_{\geq 0} : m_1 + m_2 \leq n, m_1 + 2m_2 \leq p^\alpha - 1\}.$$

Moreover, if $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, then

$$|\mathcal{F}| \leq \sum_{m=0}^{p^\alpha-1} \binom{n}{m}.$$

Proof. The proof of this theorem is similar to that of Theorem 1.

For a fixed vector $\mathbf{x} \in \Sigma$, we define the following function of the vectors $\mathbf{y} \in \Sigma$:

$$\mathcal{G}_{\mathbf{x}}(\mathbf{y}) = \frac{1}{p^\beta} \prod_{i=1}^{p^\alpha-1} (i + d - (\mathbf{x}, \mathbf{y}));$$

here $\beta \in \mathbb{Z}_{\geq 0}$ is the maximum degree of an occurrence of p in $(p^\alpha - 1)!$, i.e., β is such that $(p^\alpha - 1)!/p^\beta$ is a natural number not divisible by p .

Lemma 4. *The function $\mathcal{G}_{\mathbf{x}}(\mathbf{y})$ takes only integer values for $\mathbf{x}, \mathbf{y} \in \Sigma$, and*

$$\mathcal{G}_{\mathbf{x}}(\mathbf{y}) \equiv 0 \pmod{p} \iff (\mathbf{x}, \mathbf{y}) \not\equiv d \pmod{p^\alpha}.$$

The proof of this lemma closely follows the scheme of the proof of Lemma 2 with $\ell_1(\mathbf{x}, \mathbf{y})/2$ replaced by $(\mathbf{x}, \mathbf{y}) - d$.

Consider $\mathcal{G}_{\mathbf{x}}(\mathbf{y})$ as a polynomial in $\mathbb{Q}[y_1, \dots, y_n]$, writing out the inner product:

$$\mathcal{G}_{\mathbf{x}}(\mathbf{y}) = \frac{1}{p^\beta} \prod_{i=1}^{p^\alpha-1} (i + d - [x_1 \cdot y_1 + \dots + x_n \cdot y_n]).$$

Removing the parentheses and taking into account the fact that, for $\mathbf{y} \in \Sigma$, the equality $y_i^3 = y_i$ holds, we replace each natural degree of an occurrence of the variable $\delta \in \mathbb{Z}_{>0}$ by $2 - (\delta \bmod 2)$. We obtain the polynomials $\tilde{\mathcal{G}}_{\mathbf{x}}$:

$$\tilde{\mathcal{G}}_{\mathbf{x}}(\mathbf{y}) = \frac{1}{p^\beta} \sum_{\beta_1 + \dots + \beta_n \leq p^\alpha - 1, 0 \leq \beta_k \leq 2} \gamma_{\beta_1 \dots \beta_n} y_1^{\beta_1} \dots y_n^{\beta_n}, \quad (4.3)$$

where $\gamma_{\beta_1 \dots \beta_n} \in \mathbb{Z}$. Again note that the values of the modified polynomials $\tilde{\mathcal{G}}_{\mathbf{x}}$ at the vectors of the set Σ do not differ from the values of the original functions $\mathcal{G}_{\mathbf{x}}$. We see that the polynomial $\tilde{\mathcal{G}}_{\mathbf{x}}$ lies in some subspace $\tilde{\mathcal{G}}$ of the space $\mathbb{Q}[y_1, \dots, y_n]$ over the field $\mathbb{Q}$. The basis $\tilde{\mathcal{G}}$ is constituted by all possible monomials of the form $y_1^{\beta_1} \dots y_n^{\beta_n}$ with the following constraints on their degrees:

$$\beta_1 + \dots + \beta_n \leq p^\alpha - 1, \quad 0 \leq \beta_k \leq 2.$$

Lemma 5. *Let $\{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subset \Sigma$ be a subset of vectors such that $(\mathbf{x}_i, \mathbf{x}_j) \not\equiv d \pmod{p^\alpha}$ for all $i, j, i \neq j$. Then the polynomials $\tilde{\mathcal{G}}_{\mathbf{x}_1}, \dots, \tilde{\mathcal{G}}_{\mathbf{x}_s}$ corresponding to these vectors are linearly independent over the field $\mathbb{Q}$.*

The proof of this lemma is similar to that of Lemma 3.

Lemma 5 ensures that $|\mathcal{F}| \leq \dim \tilde{\mathcal{G}}$. Using the representation (4.3), we can estimate the dimension of the space $\tilde{\mathcal{G}}$. Let us enumerate the monomials of the set whose linear hull over $\mathbb{Q}$ includes $\tilde{\mathcal{G}}$. To do this, we first choose m_1 variables among $y_1, \dots, y_n$ that will appear to the power 1 in the monomial, and then choose m_2 variables among the remaining ones that will appear to the power 2. Of course, we must have $m_1 + m_2 \leq n$, and the constraint on the degrees now becomes the requirement $m_1 + 2m_2 \leq p^\alpha - 1$. Thus,

$$|\mathcal{F}| \leq \dim \tilde{\mathcal{G}} \leq \sum_{(m_1, m_2) \in \mathcal{A}} \binom{n}{m_1} \binom{n - m_1}{m_2},$$

where

$$\mathcal{A} = \{(m_1, m_2) : m_1, m_2 \in \mathbb{Z}_{\geq 0} : m_1 + m_2 \leq n, m_1 + 2m_2 \leq p^\alpha - 1\}.$$

In the case where $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, the identity $y_i^2 = y_i$ holds; therefore, in the representation (4.3), we obtain the more stringent constraint $0 \leq \beta_k \leq 1$. In terms of the argument above, this means that we

abandon the squares of the variables that are no longer required, thus setting $m_2 = 0$ for all summands. This immediately leads to the following refinement of the estimate:

$$|\mathcal{F}| \leq \dim \tilde{\mathcal{G}} \leq \sum_{m_1=0}^{p^\alpha-1} \binom{n}{m_1},$$

which concludes the proof of the theorem. $\square$

5. REFINEMENT OF THE DIMENSION OF SPACES OF POLYNOMIALS

We have seen above that the linear-algebraic method allows us to obtain an upper bound for the cardinality of a set of vectors in terms of the dimension of a linear space over the field $\mathbb{Q}$. It turns out that the spaces $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{G}}$ examined in the proofs of Theorems 1 and 2 can be, in a certain sense, replaced by their certain subspaces $\hat{\mathcal{P}}$ and $\hat{\mathcal{G}}$, respectively, which will lead to a strengthening of both theorems.

In this section, we describe such replacements for the cases of the Manhattan and the Euclidean metric.

5.1. The Case of the Manhattan Metric ℓ_1

Theorem 3. *Let p be a prime, and let $n \in \mathbb{N}$. Consider an arbitrary set $\Sigma \subset \{-1, 0, 1\}^n$ of vectors such that all vectors from Σ contain exactly d nonzero coordinates and a subset $\mathcal{F} \subset \Sigma$ such that, for some $\alpha \in \mathbb{N}$ and any $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{F}$, the following equivalence holds:*

$$\frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \equiv 0 \pmod{p^\alpha} \iff \mathbf{x} = \mathbf{y}.$$

Suppose that $d \geq p^\alpha - 1$. Then

$$|\mathcal{F}| \leq \binom{n}{p^\alpha - 1} 2^{p^\alpha - 1}.$$

Moreover, if $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, then

$$|\mathcal{F}| \leq \binom{n}{p^\alpha - 1}.$$

Remark. In the formulation of the theorem, we treat the quantity $\ell_1(\mathbf{x}, \mathbf{y})/2$ as an integer. We can do this because

$$\forall \mathbf{x}, \mathbf{y} \in \Sigma \quad \ell_1(\mathbf{x}, \mathbf{y}) \in 2\mathbb{Z},$$

which follows from the condition that all vectors from Σ contain the same number of nonzero coordinates (see the proof of Lemma 8 below). Thus, the assumptions of Theorem 3 are more stringent than those of Theorem 1. In Sec. 7, we shall discuss the elimination of the requirement that the distances are even.

Proof. Following the proof of Theorem 1, we again construct the polynomials

$$\tilde{\mathcal{P}}_{\mathbf{x}}(\mathbf{y}) = \frac{1}{p^\beta} \sum_{m < p^\alpha, \beta_{i_k} \in \{1, 2\}} \gamma_{i_1 \dots i_m} y_{i_1}^{\beta_{i_1}} \dots y_{i_m}^{\beta_{i_m}}, \quad (5.1)$$

where $\gamma_{i_1 \dots i_m} \in \mathbb{Z}$.

Consider the values of the following monomial at the vectors $\mathbf{y} \in \Sigma$ appearing in the sum (5.1):

$$y_{i_1}^{\beta_{i_1}} \dots y_{i_m}^{\beta_{i_m}},$$

where all β_{i_j} belong to $\{1, 2\}$. We multiply this monomial by the sum $y_1^2 + \dots + y_n^2$ of the squares of all variables. On the one hand, since there are only d nonzero coordinates in the vector $\mathbf{y} \in \Sigma$, we have

$$y_{i_1}^{\beta_{i_1}} \dots y_{i_m}^{\beta_{i_m}} (y_1^2 + \dots + y_n^2) = d y_{i_1}^{\beta_{i_1}} \dots y_{i_m}^{\beta_{i_m}}.$$

On the other hand, removing the parentheses and noting the identity $y_i^3 = y_i$, we verify that

$$y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} (y_1^2 + \cdots + y_n^2) = m y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} + y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} \sum_{j \notin \{i_1, \dots, i_m\}} y_j^2.$$

Combining the last two equalities, we obtain

$$(d - m) y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} = y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} \sum_{j \notin \{i_1, \dots, i_m\}} y_j^2.$$

Let $d \neq m$. In this case, we can divide the resulting equality by $(d - m)$:

$$y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} = \frac{1}{d - m} y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} \sum_{j \notin \{i_1, \dots, i_m\}} y_j^2. \quad (5.2)$$

Note that, for the monomial $y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}}$, we have just constructed a linear combination of monomials over the field $\mathbb{Q}$ containing a *strictly greater* number of unique variables; the values of this linear combination at $\mathbf{y} \in \Sigma$ coincide with values of the original monomial. In doing that, we have used the inequalities $d \neq m$ (the invertibility condition) and $m + 1 \leq p^\alpha - 1$ (the condition that there is at least one “free” variable).

Let us now apply the procedure described above to the monomials appearing in the expansion (5.1) whenever possible. We denote the result of the application of this operation to the polynomial $\tilde{\mathcal{P}}_{\mathbf{x}}$ by $\hat{\mathcal{P}}_{\mathbf{x}}$. The polynomial $\hat{\mathcal{P}}_{\mathbf{x}}$ possesses the following properties. First, by construction,

$$\forall \mathbf{x}, \mathbf{y} \in \Sigma \quad \tilde{\mathcal{P}}_{\mathbf{x}}(\mathbf{y}) = \hat{\mathcal{P}}_{\mathbf{x}}(\mathbf{y}).$$

Second, $\hat{\mathcal{P}}_{\mathbf{x}}$ can be expressed as a linear combination (with rational coefficients) of monomials $y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}}$ for which $m = p^\alpha - 1$. Indeed, by construction, it is required that, for the monomials appearing in the expansion, either $d = m$ or $m + 1 > p^\alpha - 1$; the assumption $d = m$ immediately leads us to $m \geq p^\alpha - 1$, while the case $m > p^\alpha - 1$ is impossible.

The following statement holds.

Lemma 6. *Let $\mathbf{x}_1, \dots, \mathbf{x}_s \in \Sigma$ be different vectors such that*

$$\forall i, j, \quad i \neq j, \quad \frac{\ell_1(\mathbf{x}_i, \mathbf{x}_j)}{2} \not\equiv 0 \pmod{p^\alpha}.$$

Then the polynomials $\hat{\mathcal{P}}_{\mathbf{x}_1}, \dots, \hat{\mathcal{P}}_{\mathbf{x}_s}$ corresponding to these vectors are linearly independent over the field $\mathbb{Q}$.

Proof. The proof does not differ from that of Lemma 3: we use the coincidence of the values of the original functions and the “modified” polynomials. $\square$

Using Lemma 6, we write $|\mathcal{F}| \leq \dim \hat{\mathcal{P}}$, where $\hat{\mathcal{P}}$ is an inclusion-minimal space containing all polynomials $\hat{\mathcal{P}}_{\mathbf{x}}, \mathbf{x} \in \Sigma$.

It only remains to find an upper bound for the dimension of $\hat{\mathcal{P}}$ over $\mathbb{Q}$. This dimension is at most the number of monomials containing strictly $p^\alpha - 1$ variables to the powers 1 and 2. This number is

$$\binom{n}{p^\alpha - 1} 2^{p^\alpha - 1}.$$

It remains to consider the case in which $\Sigma \subset \{0, 1\}^n$. In this case, $y_{i_k}^2 = y_{i_k}$; therefore, it is possible not to include the monomials containing squares of variables in $\hat{\mathcal{P}}$. This leads us to the estimate

$$\binom{n}{p^\alpha - 1},$$

which concludes the proof of the theorem. $\square$

5.2. The Case of the Euclidean Metric ℓ_2

Theorem 4. Let p be a prime, and let $n \in \mathbb{N}$. Consider an arbitrary set $\Sigma \subset \{-1, 0, 1\}^n$ of vectors such that $(\mathbf{x}, \mathbf{x}) = d$ for all $\mathbf{x} \in \Sigma$ and a subset $\mathcal{F} \subset \Sigma$ such that, for some $\alpha \in \mathbb{N}$ and any pair of different $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{F}$,

$$(\mathbf{x}, \mathbf{y}) \not\equiv d \pmod{p^\alpha},$$

where $(\mathbf{x}, \mathbf{y})$ is the Euclidean inner product of vectors in $\mathbb{R}^n$ and $d \in \mathbb{N}$ is a constant. Suppose that $d \geq p^\alpha - 1$. Then

$$|\mathcal{F}| \leq \sum_{(m_1, m_2) \in \mathcal{D}} \binom{n}{m_1} \binom{n - m_1}{m_2},$$

where

$$\mathcal{D} = \{(m_1, m_2) : m_1, m_2 \in \mathbb{Z}_{\geq 0} : m_1 + 2m_2 \in \{p^\alpha - 1, p^\alpha - 2\}\}.$$

Moreover, if $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, then

$$|\mathcal{F}| \leq \binom{n}{p^\alpha - 1}.$$

Proof. The proof is similar to that of Theorems 2 and 3.

We begin, as before, with the construction of the polynomials

$$\begin{aligned} \mathcal{G}_{\mathbf{x}}(\mathbf{y}) &= \frac{1}{p^\beta} \prod_{i=1}^{p^\alpha-1} (i + d - (\mathbf{x}, \mathbf{y})) = \frac{1}{p^\beta} \prod_{i=1}^{p^\alpha-1} (i + d - [x_1 \cdot y_1 + \cdots + x_n \cdot y_n]), \\ \tilde{\mathcal{G}}_{\mathbf{x}}(\mathbf{y}) &= \frac{1}{p^\beta} \sum_{\beta_1 + \cdots + \beta_n \leq p^\alpha-1, 0 \leq \beta_k \leq 2} \gamma_{\beta_1 \dots \beta_n} y_1^{\beta_1} \cdots y_n^{\beta_n}, \end{aligned} \quad (5.3)$$

which take coinciding values at the arguments from Σ .

Following the scheme of proof of Theorem 3, consider a monomial

$$y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}},$$

where $\beta_{i_k} \in \{1, 2\}$, appearing in the sum (5.3). Similarly, for this monomial, we construct a linear combination of monomials taking the same values at the vectors $\mathbf{y} \in \Sigma$.

Taking into account the fact that $y_1^2 + \cdots + y_n^2 = d$ and assuming that $d \neq m$, we obtain

$$y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} = \frac{1}{d - m} y_{i_1}^{\beta_{i_1}} \cdots y_{i_m}^{\beta_{i_m}} \sum_{j \notin \{i_1, \dots, i_m\}} y_j^2. \quad (5.4)$$

For the case in which $m + 1 \leq p^\alpha - 1$, the right-hand side of the resulting expansion is the sum of monomials each of which contains a strictly greater number of variables than the original one.

Using this rule, we replace all monomials appearing in the expansion $\tilde{\mathcal{G}}_{\mathbf{x}}(\mathbf{y})$ by linear combinations whenever possible: in addition to the condition $d \neq m$, we require that the replaced monomials have a free variable, i.e., $m + 1 \leq n$; we also impose the following condition on the power of each linear combination:

$$m_1 + 2m_2 + 2 \leq p^\alpha - 1,$$

where $m = m_1 + m_2$, $m_1, m_2 \in \mathbb{Z}_{\geq 0}$, are the numbers of the variables to the powers 1 and 2, respectively, appearing in the monomial. We denote the resulting polynomials by $\hat{\mathcal{G}}_{\mathbf{x}}$. By construction, we have

$$\forall \mathbf{x}, \mathbf{y} \in \Sigma \quad \tilde{\mathcal{G}}_{\mathbf{x}}(\mathbf{y}) = \hat{\mathcal{G}}_{\mathbf{x}}(\mathbf{y});$$

this allows us to obtain the following lemma, which is similar to the previous ones.

Lemma 7. *Let $\mathbf{x}_1, \dots, \mathbf{x}_s \in \Sigma$ be different vectors such that*

$$\forall i, j, \quad i \neq j, \quad (\mathbf{x}_i, \mathbf{x}_j) \not\equiv d \pmod{p^\alpha}.$$

Then the polynomials $\widehat{\mathcal{G}}_{\mathbf{x}_1}, \dots, \widehat{\mathcal{G}}_{\mathbf{x}_s}$ corresponding to these vectors are linearly independent over the field $\mathbb{Q}$.

Yet again, to estimate $|\mathcal{F}|$, it remains to estimate the dimension of the space $\widehat{\mathcal{G}}$ containing all $\widehat{\mathcal{G}}_{\mathbf{x}}$. To do this, we shall consider the process of constructing the polynomials $\widehat{\mathcal{G}}_{\mathbf{x}}$ in more detail. Let us describe cases in which the monomial cannot be expressed as a linear combination by the procedure stated above.

In view of the requirement $d \geq p^\alpha - 1$, we have the following chain of inequalities:

$$\begin{aligned} m_1 + 2m_2 + 2 \leq p^\alpha - 1 &\implies m < p^\alpha - 1 \\ &\implies m < d \implies m + 1 \leq n. \end{aligned}$$

Hence, in constructing $\widehat{\mathcal{G}}_{\mathbf{x}}$, the requirement concerning free variables is redundant, because it is included in the stronger condition on the degrees

Now assume that $m = d$. Then $m \geq p^\alpha - 1$, which is possible only for $m = p^\alpha - 1$ by the definition of the polynomials $\mathcal{G}_{\mathbf{x}}$. This condition leads to the violation of the condition on the degrees, because it implies $m_1 + 2m_2 + 2 > p^\alpha - 1$.

Thus, the impossibility of expressing the monomial as the linear combination according to (5.4) is fully described by the condition $m_1 + 2m_2 + 2 > p^\alpha - 1$, which leads to the desired estimate

$$|\mathcal{F}| \leq \sum_{(m_1, m_2) \in \mathcal{D}} \binom{n}{m_1} \binom{n - m_1}{m_2},$$

where

$$\mathcal{D} = \{(m_1, m_2) : m_1, m_2 \in \mathbb{Z}_{\geq 0} : m_1 + 2m_2 \in \{p^\alpha - 1, p^\alpha - 2\}\}.$$

The situation is significantly simplified in the “absence of minus ones”: for $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, we construct the space $\widehat{\mathcal{G}}$, excluding the monomials with squares of variables in view of $y_i^2 = y_i$. Expressing the monomials as a linear combination, we now can multiply them by $y_1 + \dots + y_n = d$ instead of using the sum of squares. This leads us to the estimate

$$|\mathcal{F}| \leq \binom{n}{p^\alpha - 1}.$$

The theorem is proved. □

6. CONSTRUCTION OF ENCLOSING SETS Σ , MAXIMAL DISTANCES, AND THE STRUCTURE OF CALCULATIONS

In this section, we shall describe a scheme allowing us to apply the above-stated theorems to obtain numerical estimates of chromatic numbers. The argument will be structured as follows.

Let a metric space $(\mathbb{R}^n, \rho)$ be given. By a *distance graph* we mean a graph $G = (V, E)$ such that each vertex from V can be identified with a point $\mathbb{R}^n$ so that the set of edges E coincides with the set of pairs of vertices some forbidden distance ρ_{crit} apart.

Given a distance graph G , we can estimate (irrespective of the parameter ρ_{crit} !) the chromatic number of the space $(\mathbb{R}^n, \rho)$:

$$\chi(\mathbb{R}^n, \rho) \geq \chi(G) \geq \frac{|V|}{\alpha(G)}, \quad (6.1)$$

where $\alpha(G)$ is the *independence number* of the graph G , i.e., the greatest possible cardinality of a set of vertices of this graph not joined by edges with one another.

Inequality (6.1) reduces the problem of estimating the chromatic number of a space from below to that of estimating the independence number of the distance graph from above. We solve the latter problem by using the theorems stated in Secs. 4 and 5.

To this end, for the metric $\rho \in \{\ell_1, \ell_2\}$, we consider three series of graphs G_1, G_2 , and G_3 , where

$$\begin{aligned} G_1(n, k_{-1}, k_0, k_1, \rho, \rho_{\text{crit}}) &= (V_1(n, k_{-1}, k_0, k_1), E_1(n, k_{-1}, k_0, k_1, \rho, \rho_{\text{crit}})), \quad k_1 \neq 0, \quad k_{-1} \neq 0, \\ V_1 &= \{\mathbf{v} = (x_1, \dots, x_n) \in \{-1, 0, 1\}^n : |\{j : x_j = i\}| = k_i, i \in \{-1, 0, 1\}\}, \\ G_2(n, d, \rho, \rho_{\text{crit}}) &= (V_2(n, d), E_2(n, d, \rho, \rho_{\text{crit}})), \\ V_2 &= \{\mathbf{v} = (x_1, \dots, x_n) \in \{-1, 0, 1\}^n : |\{j : x_j = 1\}| + |\{j : x_j = -1\}| = d\}, \\ G_3(n, k_1, \rho, \rho_{\text{crit}}) &= (V_3(n, k_1), E_3(n, d, \rho, \rho_{\text{crit}})), \\ V_3 &= \{\mathbf{v} = (x_1, \dots, x_n) \in \{0, 1\}^n : |\{j : x_j = 1\}| = k_1\} \\ E_i &= \{(\mathbf{v}_1, \mathbf{v}_2) \in V_i \times V_i : \rho(\mathbf{v}_1, \mathbf{v}_2) = \rho_{\text{crit}}\}, \quad i = 1, 2, 3. \end{aligned}$$

Embedding G_1, G_2 , and G_3 in $\mathbb{R}^n$ in a natural way, we can verify that the constructed graphs are distance graphs.

Note that the graphs of the series G_1 and G_3 can, in fact, be embedded in the space $\mathbb{R}^{n-1}$ as well, because the vertices of these graphs satisfy the linear relations $x_1 + \dots + x_n = k_1 - k_{-1}$ and $x_1 + \dots + x_n = d$, respectively.

For the enclosing sets Σ we take the vertex sets V_i . Note that we can do so in order to apply both theorems of Sec. 4 and those of Sec. 5, because, for each of the sets V_i , its vectors have equal numbers of nonzero coordinates.

Subsequent considerations will depend on the choice of the metric ρ .

6.1. The Case of the Manhattan Metric: Calculation of Maximum Distances

Lemma 8. *Let $\mathbf{u}, \mathbf{v} \in V_i, i = 1, 2, 3$. Then $\ell_1(\mathbf{u}, \mathbf{v})$ is even.*

Proof. Let $\mathbf{u} = (u_1, \dots, u_n)$, and let $\mathbf{v} = (v_1, \dots, v_n)$. As we know, these vectors contain the same number d of nonzero coordinates. Suppose that m coordinates are nonzero for both vectors simultaneously. Each coordinate i makes the contribution $|u_i - v_i|$, equal to 0 or 2, to the sum. We obtain

$$\ell_1(\mathbf{u}, \mathbf{v}) = \sum_i |u_i - v_i| = \sum_{i: u_i, v_i \neq 0} |u_i - v_i| + \sum_{i: u_i=0, v_i \neq 0} |u_i - v_i| + \sum_{i: u_i \neq 0, v_i=0} |u_i - v_i|.$$

Here the first of the three sums is even and each of the remaining ones is equal to $d - m$; hence $\ell_1(\mathbf{u}, \mathbf{v})$ is also even. $\square$

Lemma 9. *The maximum ℓ_1 -distance between two vectors in the space $V_1(n, k_{-1}, k_0, k_1)$ is*

$$\begin{cases} 4k_1 + 2k_0, & k_1 < k_{-1} - k_0, \\ 2k_1 + 2k_{-1}, & k_{-1} - k_0 \leq k_1 < k_{-1} + k_0, \\ 4k_{-1} + 2k_0, & k_{-1} + k_0 \leq k_1. \end{cases}$$

Proof. It follows from symmetry considerations that, for any vector $\mathbf{u} \in V_1$, there exists a vector $\mathbf{v} \in V_1$ at the required maximal distance from $\mathbf{u}$. We shall assume that, in the vector $\mathbf{v}$, first, 1's follow one another, next 0's, and then -1's. The distance between the vectors $\mathbf{u}$ and $\mathbf{v}$ is expressed by a coordinate-wise sum, and the contribution of each coordinate i to this sum is

$$\begin{cases} 2 & \text{if } u_i = -v_i, \\ 1 & \text{if precisely one of the numbers } u_i \text{ and } v_i \text{ is zero,} \\ 0 & \text{in the case } u_i = v_i. \end{cases}$$

Therefore, the maximum of the distance will be attained if the vector $\mathbf{v}$ is constructed by “overturning” the vector $\mathbf{u}$: $v_i = u_{n-i+1}$.

To complete the proof of the lemma, it is necessary to consider several cases characterized by different relations between k_1 and the quantities k_{-1} , $k_{-1} - k_0$, and $k_{-1} + k_0$. For clarity, we shall depict the vector and the corresponding “inverted” vector in each of the cases.

Case 1: $k_{-1} \leq k_1 < k_{-1} + k_0$.

1	0	-1
-1	0	1

Case 2: $k_{-1} + k_0 \leq k_1$.

1	0	-1
-1	0	1

Case 3: $k_{-1} - k_0 \leq k_1 < k_{-1}$.

1	0	-1
-1	0	1

Case 4: $k_1 < k_{-1} - k_0$.

1	0	-1
-1	0	1

□

Lemma 10. *The maximum ℓ_1 -distance between two vectors from the space $V_2(n, d)$ is $2d$.*

Proof. The maximum distance is attained for oppositely directed vectors. □

Lemma 11. *The maximum ℓ_1 -distance between two vectors from the space $V_3(n, k_1)$ is*

$$\begin{cases} 2k_1, & 2k_1 \leq n, \\ 2(n - k_1), & 2k_1 > n. \end{cases}$$

Proof. The proof is obvious, □

6.2. The Case of the Manhattan Metric: Enumeration of Parameters and Numerical Results

In our calculations, we considered the sequences of graphs

$$G_1(n + 1, k_{-1}, k_0, k_1, \ell_1, \ell_1^{\text{crit}}), \quad G_2(n, d, \ell_1, \ell_1^{\text{crit}}), \quad G_3(n + 1, k_1, \ell_1, \ell_1^{\text{crit}}).$$

(Here it is important that, to estimate the chromatic number of an n -dimensional space, we may use the graphs $G_1(n + 1, \dots)$ and $G_3(n + 1, \dots)$ to obtain the sharpest estimates.) For the examined values of n , we enumerated all possible values of k_{-1} , k_0 , and k_1 for G_1 and G_3 , as well as all possible d for G_2 . Fixing the given parameters, for each of the resulting graphs $G = (V, E)$, we obtained the maximum possible distance ℓ_1^{max} between two vectors from V by using the lemmas stated above. After that, for each graph, we chose the minimum¹ primary number $q = p^\alpha$ (here p is a prime and $\alpha \in \mathbb{N}$) satisfying the condition $4q > \ell_1^{\text{max}}$. Then we chose the forbidden distance as $\ell_1^{\text{crit}} = 2q$, and for Σ , we chose one of the sets V_1 , V_2 , and V_3 . Further, to estimate the independence numbers $\alpha(G_i)$, we applied Theorems 1

¹As can be seen, the estimate of the independence number provided by our theorems is the sharper the smaller the chosen q , other things being equal.

and 3, using the following consideration: any independence set of our graphs may be regarded as the set $\mathcal{F}$ from the assumptions of the theorems in view of the fact that

$$\begin{aligned} \forall \mathbf{x}, \mathbf{y} \in \Sigma \quad \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \equiv 0 \pmod{q} &\iff \frac{\ell_1(\mathbf{x}, \mathbf{y})}{2} \in \{0, q\} \\ &\iff \ell_1(\mathbf{x}, \mathbf{y}) \in \{0, 2q\}. \end{aligned}$$

Finally, for each examined graph, using the upper bound for its independence number and following (6.1), we found a lower bound for $\chi(\mathbb{R}^n, \ell_1)$. Below we present the obtained numerical results (see Tables 3 and 4). The calculations were performed by a program in **Python 2.7** [53].

For each dimension, calculations on graphs of the three types G_1 , G_2 , and G_3 were performed. In each case, we used both Theorem 1 and its strengthened version, Theorem 3, so that the table consists of two parts, the “naïve” and the “refined” one. The best result for each dimension is highlighted in **bold**, independently for two theorems. Also we give the values of the parameters of the distance graphs for which the strongest result was obtained.

Table 3

$n \downarrow$	by Theorem 1					
	G_1		G_2		G_3	
graph $\rightarrow$	$\chi \geq$	(k_{-1}, k_0, k_1)	$\chi \geq$	d	$\chi \geq$	k_1
4	3	(2, 2, 1)	4	3	2	3
5	5	(2, 3, 1)	8	3	3	4
6	7	(2, 4, 1)	13	3	5	4
7	10	(2, 5, 1)	19	3	7	5
8	14	(2, 6, 1)	27	3	9	6
9	18	(2, 7, 1)	36	3	11	7
10	22	(2, 8, 1)	46	3	14	8
11	28	(3, 7, 2)	61	5	17	9
12	38	(3, 8, 2)	88	5	21	10
13	51	(3, 9, 2)	122	5	25	11
14	67	(3, 10, 2)	164	5	29	12
15	86	(3, 11, 2)	214	5	33	13
16	114	(4, 10, 3)	294	7	41	12
17	156	(4, 11, 3)	414	7	50	13
18	209	(4, 12, 3)	568	7	61	14
19	274	(4, 13, 3)	762	7	74	15
20	354	(4, 14, 3)	1001	7	88	16
21	481	(5, 13, 4)	1403	9	104	17
22	657	(5, 14, 4)	1955	9	122	18
23	880	(5, 15, 4)	2667	9	149	17
24	1160	(5, 16, 4)	3574	9	184	18

Table 4

$n \downarrow$	by Theorem 3					
graph $\rightarrow$	G_1		G_2	G_3		
	$\chi \geq$	(k_{-1}, k_0, k_1)	$\chi \geq$	d	$\chi \geq$	k_1
4	3	(2, 2, 1)	4	3	2	3
5	5	(2, 3, 1)	8	3	4	3
6	8	(2, 4, 1)	14	3	5	4
7	11	(2, 5, 1)	20	3	7	5
8	14	(2, 6, 1)	28	3	10	6
9	18	(2, 7, 1)	38	3	12	7
10	23	(2, 8, 1)	48	3	15	8
11	30	(3, 7, 2)	68	5	19	9
12	42	(3, 8, 2)	96	5	22	10
13	55	(3, 9, 2)	132	5	26	11
14	72	(3, 10, 2)	176	5	31	12
15	91	(3, 11, 2)	229	5	37	11
16	126	(4, 10, 3)	327	7	46	12
17	171	(4, 11, 3)	458	7	56	13
18	228	(4, 12, 3)	624	7	68	14
19	298	(4, 13, 3)	832	7	82	15
20	387	(5, 12, 4)	1110	9	97	16
21	536	(5, 13, 4)	1572	9	114	17
22	727	(5, 14, 4)	2176	9	139	16
23	969	(5, 15, 4)	2954	9	171	17
24	1272	(5, 16, 4)	3938	9	209	18

Note that, for all $n = 4, \dots, 24$, the strongest result was obtained for graphs of type G_2 , the presence of -1 's is favorable; moreover, it turns out to be more optimal to fix the *total* number of nonzero coordinates in the vectors from Σ .

6.3. The Case of the Euclidean Metric: Calculation of Maximal Distances

Lemma 12. *Let $\mathbf{u}, \mathbf{v} \in V_i, i = 1, 2, 3$. Then $\ell_2^2(\mathbf{u}, \mathbf{v})$ is an even number.*

Proof. We write out the squared distance between two vectors as follows:

$$\ell_2^2(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_2^2 = (\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = (\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v}) - 2(\mathbf{u}, \mathbf{v}).$$

It remains to see that the first two summands are equal, because the number of nonzero coordinates is fixed. $\square$

Lemma 13. *The maximum squared ℓ_2 -distance between two vectors in $V_1(n, k_{-1}, k_0, k_1)$ is*

$$\begin{cases} 8k_1 + 2k_0, & k_1 < k_{-1} - k_0, \\ 6k_1 + 2k_{-1}, & k_{-1} - k_0 \leq k_1 < k_{-1}, \\ 6k_{-1} + 2k_1, & k_{-1} \leq k_1 < k_{-1} + k_0, \\ 8k_{-1} + 2k_0, & k_{-1} + k_0 \leq k_1. \end{cases}$$

Proof. It suffices to consider a vector with ordered coordinates and its “inverted” analog. The proof reduces to the study of the same cases that were illustrated in the proof of Lemma 9. $\square$

Lemma 14. *The maximum squared ℓ_2 -distance between two vectors in the space $V_2(n, d)$ is $4d$.*

Proof. In this case, the maximum distance is attained for oppositely directed vectors. $\square$

Lemma 15. *The maximum squared ℓ_2 -distance between two vectors in the space $V_3(n, k_1)$ is*

$$\begin{cases} 2k_1, & 2k_1 \leq n, \\ 2(n - k_1), & 2k_1 > n. \end{cases}$$

6.4. The Case of the Euclidean Metric: Enumeration of Parameters and Numerical Results

As in the case of the metric ℓ_1 , here we considered all possible sets of parameters k_{-1}, k_0, k_1, d for graphs from the series

$$G_1(n+1, k_{-1}, k_0, k_1, \ell_2, \ell_2^{\text{crit}}), \quad G_2(n, d, \ell_2, \ell_2^{\text{crit}}), \quad G_3(n+1, k_1, \ell_2, \ell_2^{\text{crit}})$$

for small n . After this, fixing the choice of parameters, for each resulting graph $G = (V, E)$, we calculated the maximum possible squared distance $(\ell_2^{\max})^2$ between two vectors in V , using Lemmas 13, 14, and 15. Further, we chose the minimum primary number $q = p^\alpha$ satisfying the condition

$$4q > (\ell_2^{\max})^2.$$

Again taking $\Sigma = V$, we then chose the forbidden distance

$$\ell_2^{\text{crit}} = \sqrt{2q}.$$

Note that such a choice corresponds to the exclusion of inner products equal to the number of nonzero elements in the vector-vertices of the constructed graph. Thus, we can apply Theorems 2 and 4; we chose $d = k_{-1} + k_1$ and $d = k_1$ for G_1 and G_3 , respectively, in the notation of these theorems. Indeed, in the general case, denoting the number of nonzero coordinates of vectors by d , for arbitrary $\mathbf{u}, \mathbf{v} \in V$, we can obtain

$$\begin{aligned} \ell_2^2(\mathbf{u}, \mathbf{v}) &= (\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = (\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v}) - 2(\mathbf{u}, \mathbf{v}) = 2d - 2(\mathbf{u}, \mathbf{v}), \\ (\mathbf{u}, \mathbf{v}) &= d - \frac{\ell_2^2(\mathbf{u}, \mathbf{v})}{2}. \end{aligned}$$

In view of the choice of q , for $\mathbf{u} \neq \mathbf{v}$, we then have

$$(\mathbf{u}, \mathbf{v}) \equiv d \pmod{q} \quad \Longleftrightarrow \quad \frac{\ell_2^2(\mathbf{u}, \mathbf{v})}{3} \equiv 0 \pmod{q} \quad \Longleftrightarrow \quad \ell_2^2(\mathbf{u}, \mathbf{v}) = 2q.$$

Thus, we have constructed a correspondence between the sets $\mathcal{F}$ from Theorems 2 and 4 and the independence sets of the distance graphs under examination. The resulting upper bounds for the independence numbers, just as in the case of the Manhattan metric, were used to obtain lower bounds for $\chi(\mathbb{R}^n, \ell_2)$ for small n .

Below we present the tables of the obtained numerical results similar to the table given earlier for ℓ_1 (see Tables 4 and 5).

Table 5

$n, \downarrow$	by Theorem 2					
graph $\rightarrow$	G_1		G_2		G_3	
	$\chi \geq$	(k_{-1}, k_0, k_1)	$\chi \geq$	d	$\chi \geq$	k_1
4	2	(2, 2, 1)	2	3	2	3
5	3	(2, 3, 1)	2	3	3	4
6	3	(2, 4, 1)	3	3	5	4
7	4	(2, 5, 1)	3	4	7	5
8	5	(2, 6, 1)	3	4	9	6
9	6	(2, 7, 1)	4	4	11	7
10	7	(4, 6, 1)	4	4	14	8
11	9	(4, 7, 1)	5	4	17	9
12	12	(4, 8, 1)	5	6	21	10
13	16	(4, 9, 1)	6	6	25	11
14	19	(4, 10, 1)	7	6	29	12
15	23	(4, 11, 1)	8	7	33	13
16	28	(4, 12, 1)	9	7	41	12
17	33	(4, 13, 1)	11	7	50	13
18	42	(6, 12, 1)	12	8	61	14
19	54	(6, 13, 1)	14	8	74	15
20	67	(6, 14, 1)	16	8	88	16
21	83	(6, 15, 1)	18	8	104	17
22	101	(6, 16, 1)	20	10	122	18
23	122	(6, 17, 1)	24	10	149	17
24	146	(6, 18, 1)	28	10	184	18
25	173	(6, 19, 1)	32	10	223	19
26	203	(6, 20, 1)	36	10	269	20
27	236	(6, 21, 1)	41	12	322	21
28	273	(6, 22, 1)	48	12	382	22
29	337	(10, 19, 1)	56	12	450	23
30	436	(10, 20, 1)	65	12	553	22
31	557	(10, 21, 1)	74	12	677	23
32	704	(10, 22, 1)	83	12	822	24
33	881	(10, 23, 1)	94	15	991	25
34	1093	(10, 24, 1)	112	15	1186	26
35	1344	(10, 25, 1)	131	15	1412	27
36	1641	(10, 26, 1)	152	15	1670	28

37	1990	(10, 27, 1)	175	15	1965	29
38	2395	(10, 28, 1)	202	16	2300	30
39	2866	(10, 29, 1)	234	16	2679	31
40	3475	(11, 28, 2)	268	16	3289	28
41	4206	(11, 29, 2)	305	18	4103	29
42	5155	(12, 30, 1)	357	18	5084	30
43	6293	(12, 31, 1)	414	18	6259	31
44	7636	(12, 32, 1)	478	18	7660	32
45	9213	(12, 33, 1)	547	18	9322	33
46	11057	(12, 34, 1)	623	18	11284	34
47	13505	(15, 31, 2)	705	18	13590	35
48	17062	(15, 32, 2)	794	18	16288	36
49	21407	(15, 33, 2)	933	22	19434	37
50	26687	(15, 34, 2)	1101	22	23377	36
51	33065	(15, 35, 2)	1291	22	28538	37
52	40731	(15, 36, 2)	1503	22	34669	38
53	49898	(15, 37, 2)	1739	22	41924	39
54	60809	(15, 38, 2)	2000	22	50473	40
55	73737	(15, 39, 2)	2289	22	60511	41
56	90601	(16, 38, 3)	2655	24	72254	42
57	110839	(16, 39, 3)	3084	24	87296	41
58	134921	(16, 40, 3)	3563	24	106359	42
59	163451	(16, 41, 3)	4095	24	129037	43
60	197113	(16, 42, 3)	4683	24	155914	44
61	236673	(16, 43, 3)	5456	26	187656	45
62	288637	(19, 40, 4)	6327	26	225017	46
63	361244	(19, 41, 4)	7300	26	268849	47
64	449736	(19, 42, 4)	8383	26	320112	48
65	557086	(19, 43, 4)	9634	28	379884	49
66	686734	(19, 44, 4)	11209	28	449379	50
67	842643	(19, 45, 4)	12978	28	538550	47
68	1029368	(19, 46, 4)	14957	28	664098	48
69	1286364	(21, 45, 4)	17162	28	815709	49

Table 6

$n, \downarrow$	by Theorem 4					
graph $\rightarrow$	G_1		G_2		G_3	
	$\chi \geq$	(k_{-1}, k_0, k_1)	$\chi \geq$	d	$\chi \geq$	k_1
4	2	(2, 2, 1)	2	3	2	3
5	3	(2, 3, 1)	2	4	4	3
6	3	(2, 4, 1)	3	3	5	4
7	4	(2, 5, 1)	3	4	7	5
8	5	(2, 6, 1)	4	3	10	6
9	6	(2, 7, 1)	4	4	12	7
10	7	(4, 6, 1)	5	4	15	8
11	10	(4, 7, 1)	5	6	19	9
12	13	(4, 8, 1)	6	6	22	10
13	16	(4, 9, 1)	7	6	26	11
14	20	(4, 10, 1)	8	6	31	12
15	24	(4, 11, 1)	9	7	37	11
16	28	(4, 12, 1)	10	7	46	12
17	34	(4, 13, 1)	12	7	56	13
18	43	(6, 12, 1)	13	8	68	14
19	55	(6, 13, 1)	15	8	82	15
20	69	(6, 14, 1)	18	8	97	16
21	85	(6, 15, 1)	20	8	114	17
22	103	(6, 16, 1)	23	10	139	16
23	124	(6, 17, 1)	27	10	171	17
24	148	(6, 18, 1)	31	10	209	18
25	175	(6, 19, 1)	35	10	253	19
26	206	(6, 20, 1)	40	12	304	20
27	239	(6, 21, 1)	47	12	362	21
28	276	(9, 18, 2)	55	12	428	22
29	350	(9, 19, 2)	63	12	523	21
30	446	(10, 20, 1)	72	12	641	22
31	570	(10, 21, 1)	81	12	780	23
32	719	(10, 22, 1)	91	15	943	24
33	899	(10, 23, 1)	109	15	1131	25
34	1114	(10, 24, 1)	128	15	1349	26
35	1370	(10, 25, 1)	149	15	1599	27
36	1670	(10, 26, 1)	172	15	1884	28

37	2023	(10, 27, 1)	198	16	2209	29
38	2434	(10, 28, 1)	229	16	2577	30
39	2931	(11, 27, 2)	263	16	3135	27
40	3567	(11, 28, 2)	300	16	3919	28
41	4313	(11, 29, 2)	349	18	4865	29
42	5250	(12, 30, 1)	406	18	6000	30
43	6404	(12, 31, 1)	469	18	7355	31
44	7765	(12, 32, 1)	538	18	8964	32
45	9363	(12, 33, 1)	613	18	10865	33
46	11230	(12, 34, 1)	694	18	13102	34
47	13924	(15, 31, 2)	782	18	15722	35
48	17570	(15, 32, 2)	913	22	18779	36
49	22021	(15, 33, 2)	1080	22	22535	35
50	27424	(15, 34, 2)	1267	22	27542	36
51	33946	(15, 35, 2)	1477	22	33497	37
52	41778	(15, 36, 2)	1710	22	40549	38
53	51137	(15, 37, 2)	1969	22	48867	39
54	62267	(15, 38, 2)	2255	22	58641	40
55	76133	(16, 37, 3)	2610	24	70083	41
56	93531	(16, 38, 3)	3035	24	84511	40
57	114308	(16, 39, 3)	3509	24	103062	41
58	139008	(16, 40, 3)	4036	24	125147	42
59	168248	(16, 41, 3)	4619	24	151340	43
60	202720	(16, 42, 3)	5373	26	182296	44
61	243202	(16, 43, 3)	6235	26	218756	45
62	300723	(19, 40, 4)	7199	26	261555	46
63	375921	(19, 41, 4)	8272	26	311640	47
64	467474	(19, 42, 4)	9491	28	370073	48
65	578426	(19, 43, 4)	11051	28	438045	49
66	712295	(19, 44, 4)	12803	28	523111	46
67	873132	(19, 45, 4)	14765	28	645541	47
68	1067129	(21, 44, 4)	16951	28	793477	48
69	1336483	(21, 45, 4)	19546	30	971604	49

Here the picture is much more interesting than in the case of the Manhattan metric. All the theorems work, i.e., for different dimensions, different ideas turn out to be most favorable. Certainly, the refinement

of the estimate related to the presence of -1 's among the coordinates of graph vertices manifests itself as the dimension increases, because the best asymptotic results were obtained by using such graphs. However, estimates obtained without -1 's dominate for a very long time. It is also worth noting that, for the Euclidean metric, it is more useful to fix the number of coordinates of each quantity separately, while for the Manhattan metric, it is optimal to fix the total number of nonnegative coordinates.

7. REMARKS ON THE POSSIBILITY OF ELIMINATING THE EVENNESS REQUIREMENT ON THE DISTANCES

In Theorem 1 stated above, we imposed the evenness requirement on the distances between two elements of the set Σ . In Theorem 3, the distances were even automatically. This condition was needed in Sec. 6. It is necessary to note that we can do without the explicit requirement for the evenness of distances, obtaining the following theorems.

Theorem 5. *Let p be a prime, and let $n \in \mathbb{N}$. Consider an arbitrary set of vectors $\Sigma \subset \{-1, 0, 1\}^n$ and a subset $\mathcal{F} \subset \Sigma$ such that, for some $\alpha \in \mathbb{N}$ and any pair of $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{F}$, the following equivalence holds:*

$$\ell_1(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod{p^\alpha} \iff \mathbf{x} = \mathbf{y};$$

suppose that $n \geq p^\alpha - 1$. Then

$$|\mathcal{F}| \leq \sum_{m=0}^{p^\alpha-1} 2^m \binom{n}{m}.$$

Moreover, if $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, then

$$|\mathcal{F}| \leq \sum_{m=0}^{p^\alpha-1} \binom{n}{m}.$$

Theorem 6. *Let p be a prime, and let $n \in \mathbb{N}$. Consider an arbitrary set of vectors $\Sigma \subset \{-1, 0, 1\}^n$ such that all vectors from Σ contain exactly d nonzero coordinates and a subset $\mathcal{F} \subset \Sigma$ such that, for some $\alpha \in \mathbb{N}$ and any $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{F}$, the following equivalence holds:*

$$\ell_1(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod{p^\alpha} \iff \mathbf{x} = \mathbf{y};$$

suppose that $d \geq p^\alpha - 1$. Then

$$|\mathcal{F}| \leq \binom{n}{p^\alpha - 1} 2^{p^\alpha-1}.$$

Moreover, if $\mathcal{F} \subset \Sigma \subset \{0, 1\}^n$, then

$$|\mathcal{F}| \leq \binom{n}{p^\alpha - 1}.$$

We do not give the proofs of these theorems, because they are completely similar to those of Theorems 1 and 3, respectively. The main difference is that, in constructing polynomials similar to $\mathcal{P}_{\mathbf{x}}(\mathbf{y})$, it is necessary to consider the distance $\ell_1(\mathbf{x}, \mathbf{y})$ instead of $\ell_1(\mathbf{x}, \mathbf{y})/2$.

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