Sum of the Inverse Catalan Numbers

Def 0.1 $C_n = \frac{1}{n+1}\binom{2n}{n}$.

We show

$$\sum_{n=0}^{\infty} \frac{1}{C_n} = 2 + \frac{4\pi\sqrt{3}}{27} \approx 2.806133050770763489.$$

1 Needed Lemmas

Lemma 1.1

1. For all $a, b \geq 1$,

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

2.

$$\int_0^1 t^{n-1}(1-t)^n dt = \frac{(n-1)!n!}{(2n)!}$$

(This follows from Part 1.)

Proof:
We prove this by induction on b.
Base Case: Assume $a \geq 1$ and $b = 1$.

$$\int_0^1 t^{a-1} dt = \frac{(a-1)!(1-1)!}{(a+1-1)!} = \frac{(a-1)!}{a!} = \frac{1}{a}.$$

This integral is easily seen to be true.
IH (The $b-1$ case). For all $a \geq 1$, $\int_0^1 t^{a-1}(1-t)^{b-2} dt = \frac{(a-1)!(b-2)!}{(a+b-2)!}$.
We will need this in the case of $a+1$ so we state that here:

$$\int_0^1 t^a (1-t)^{b-2} dt = \frac{a!(b-2)!}{(a+b-1)!}$$

IS We can assume $b \geq 2$.

We look at:

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt$$

We do integration by parts.

$u = (1-t)^{b-1}$ so $du = -(b-1)(1-t)^{-2}dt$.

$dv = t^{a-1}dt$ so $v = \frac{t^a}{a}$.

Note that uv is 0 at both $t = 0$ and $t = 1$ so $uv\Big|_0^1 = 0$.

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = -\int_0^1 -\frac{1}{a}t^a(b-1)(1-t)^{b-2}dt = \frac{b-1}{a}\int_0^1 t^a(1-t)^{b-2}dt.$$

By the IH we know

$$\int_0^1 t^a(1-t)^{b-2}dt = \frac{a!(b-2)!}{(a+b-1)!}$$

Hence we have

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{b-1}{a}\frac{a!(b-2)!}{(a+b-1)!} = \frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

∎

Lemma 1.2

1. $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$.

2. $\sum_{i=1}^{\infty} i^2x^i = \frac{x(1+x)}{(1-x)^3}$.

3. $\sum_{i=1}^{\infty} i(i+1)x^i = \frac{2x}{(1-x)^3}$.

Proof:
1)

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

Take the deriviative of both sides.

$$\sum_{i=0}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}$$

The first term of the sum is 0 so it can be omitted. Also we multiply both sides by x.

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$$

2)

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$$

Take the derivative of both sides.

$$\sum_{i=1}^{\infty} i^2x^{i-1} = \frac{1+x}{(1-x)^3}$$

Multiply both sides by x

$$\sum_{i=1}^{\infty} i^2x^i = \frac{x(1+x)}{(1-x)^3}$$

3)

$$\sum_{i=1}^{\infty} i(i+1)x^i = \sum_{i=1}^{\infty} i^2 x^i + \sum_{i=1}^{\infty} ix^i$$

$$= \frac{x}{(1-x)^2} + \frac{x(1+x)}{(1-x)^3} = \frac{x(1-x)}{(1-x)^3} + \frac{x(1+x)}{(1-x)^3} = \frac{2x}{(1-x)^3}.$$

∎

Lemma 1.3

1. $1 + \tan^2\theta = \sec^2\theta$
2. $\frac{1}{(1+\tan^2\theta)^3} = \cos^6\theta$ *(This follows from Part 1.)*

Proof:

$\sin^2\theta + \cos^2\theta = 1$
Divide by $\cos^2\theta$.
$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \sec^2\theta$
$\tan^2\theta + 1 = \sec^2\theta$. ∎

2 Main Theorem

Theorem 2.1

$$\sum_{n=0}^{\infty} \frac{1}{C_n} = 2 + \frac{4\pi\sqrt{3}}{27} \approx 2.806133050770763489.$$

Proof:

For $n \geq 1$,

$$\frac{1}{C_n} = \frac{(n+1)(n!)^2}{(2n)!}.$$

By Lemma 1.1.2:

$$\int_0^1 t^{n-1}(1-t)^n \, dt = \frac{(n-1)!n!}{(2n)!}.$$

Hence

$$\frac{1}{C_n} = n(n+1) \int_0^1 t^{n-1}(1-t)^n \, dt.$$

so

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = \int_0^1 \frac{1}{t} \sum_{n=1}^{\infty} n(n+1)(t(1-t))^n \, dt.$$

By Lemma 1.2.2:

$$\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3},$$

Hence we get

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = \int_0^1 \frac{2(1-t)}{(1-t+t^2)^3} \, dt = \int_0^1 \frac{dt}{(t^2-t+1)^3}.$$

Let $t = x + B$ where we determine B later

$$(x+B)^2-(x+B)+1 = x^2+2Bx+B^2-x-B+1 = x^2+(2B-1)x+(B^2-B+1)$$

We can choose $B = \frac{1}{2}$ so that coefficent of x is 0.
$t = x + \frac{1}{2}$.
$dt = dx$.
$x = t - \frac{1}{2}$.
So

$$\int_0^1 \frac{dt}{(t^2 - t + 1)^3} = \int_{-1/2}^{1/2} \frac{dx}{(x^2 + 3/4)^3} dx$$

We want x^2 to have coefficient $3/4$. We set
$x = \frac{\sqrt{3}}{2} y$.
$dx = \frac{\sqrt{3}}{2} dy$.
$y = \frac{2x}{\sqrt{3}}$.
When $x = \frac{1}{2}$, $y = \frac{1}{\sqrt{3}}$.
When $x = -\frac{1}{2}$, $y = -\frac{1}{\sqrt{3}}$.
So

$$\int_{-1/2}^{1/2} \frac{dx}{(x^2 + 3/4)^3} = \frac{\sqrt{3}}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dy}{((3/4)y^2 + 3/4)^3} = \frac{32\sqrt{3}}{27} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dy}{(y^2 + 1)^3}$$

RECAP. We have:

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = \frac{32\sqrt{3}}{27} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{(1 + x^2)^3}.$$

Letting $x = \tan\theta$ and using Lemma 1.3.2 we get:

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{(1 + x^2)^3} = \int_{-\pi/6}^{\pi/6} \cos^4\theta \, d\theta = \frac{\pi}{8} + \frac{9\sqrt{3}}{32}.$$

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Therefore

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = 1 + \frac{4\pi\sqrt{3}}{27},$$

and since $C_0 = 1$,

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{C_n} = 2 + \frac{4\pi\sqrt{3}}{27}.}$$

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