Four-Coloring Beyond the Plane:
An Update on Bounded Genus and Bounded Crossing Number

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Abstract

In the June 2022 installment of this column, Gasarch, Hayes, Ostuni, and Park asked for the complexity of deciding whether $\chi(G) \leq r$ for graphs of bounded genus or bounded crossing number, and observed that the only unresolved cases are $r = 4$ with genus or crossing number at least one. We report on progress since then, most notably a 2025 paper of Dvořák, Lidický, and Mohar attacking the crossing-number-one case, explain the structural reasons the problems resist NP-hardness proofs, present a small computational experiment quantifying how badly a single crossing damages the Kempe-chain reducibility machinery underlying the Four Color Theorem, and pose several new open problems, including parameterized and quantitative versions of the original questions.

1 Introduction

The Four Color Theorem makes 4-colorability of planar graphs trivial as a decision problem, while 3-colorability of planar graphs is NP-complete. What happens to the 4-coloring question when we leave the plane, but only barely—graphs of genus one, or graphs drawable with a single crossing? Remarkably, the complexity of these problems is open, and seems not to have been asked at all until recently: Gasarch raised the question on the Computational Complexity blog in November 2021 [10], it was developed in the June 2022 installment of this column by Gasarch, Hayes, Ostuni, and Park [11], and the genus version appears as Problem 8.4.10 in Mohar and Thomassen's monograph [15]. As we recall in Section 2, for every other number of colors the answer is known, for every fixed genus and crossing number; the case of four colors is precisely where the Four Color Theorem's reach ends and our ignorance begins.[1]

This note is a follow-up. We report on progress since 2022, most notably a 2025 paper of Dvořák, Lidický, and Mohar [7] that mounts a direct attack on the one-crossing case (Section 3). We then explain, via a simple identification argument, the structural reason NP-hardness proofs have not materialized: in any non-4-colorable graph, every crossing must be locked inside a K_4 and so carries only a bounded amount of information (Section 4). In Section 5 we present a small computational experiment on the Birkhoff diamond quantifying how thoroughly a single crossing destroys the Kempe-chain reducibility machinery that proved the Four Color Theorem, which explains why the program of [7] must rebuild that machinery rather than transplant it. Section 6 poses open problems, including parameterized and quantitative refinements of the original questions that we believe are well suited to readers of this column.

[1]Point 2(a) of [10] laments how hard it is to determine who-asked-what-when for open problems. We are happy to report that, for this problem, citing the blog post settles the provenance question the post itself raised.

2 Background and known results

All graphs are finite, simple, and undirected. The *chromatic number* $\chi(G)$ is the least r such that G has a proper r-coloring. The *genus* $g(G)$ is the least g such that G embeds in the orientable surface of genus g, and the *crossing number* $\mathrm{cr}(G)$ is the least c such that G can be drawn in the plane with c pairwise crossings of edges. Planar graphs are exactly the graphs with $g(G) = 0$, equivalently $\mathrm{cr}(G) = 0$. Note $g(G) \le \mathrm{cr}(G)$: given a drawing with c crossings, adding a handle at each crossing yields an embedding.

For fixed r, g, and c, consider the languages

$$\mathrm{COL}^{g}_{r,g} = \{G : \chi(G) \le r \text{ and } g(G) \le g\}, \qquad \mathrm{COL}^{\mathrm{cr}}_{r,c} = \{G : \chi(G) \le r \text{ and } \mathrm{cr}(G) \le c\}.$$

Since testing $g(G) \le g$ is linear-time for fixed g [14], and testing $\mathrm{cr}(G) \le c$ is linear-time for fixed c [13], all these languages lie in NP, and their complexity is really that of the promise problem: given a graph of genus at most g (crossing number at most c), decide r-colorability. (By contrast, computing the crossing number exactly is NP-hard even for graphs one edge away from planar [4]; fixing c is what makes the topological part easy.)

The column of Gasarch, Hayes, Ostuni, and Park [11] assembled the known results, which we summarize. For $r \le 2$ the problems are in P (bipartiteness). For $r = 3$ they are NP-complete for every $g \ge 0$ and $c \ge 0$: planar graphs belong to every one of these classes, and planar 3-colorability is NP-complete [9]. For $r = 4$ and $g = c = 0$ the problem is in P, indeed trivial, by the Four Color Theorem [1, 2, 17]; a coloring can even be found in quadratic time [16]. For $r \ge 5$ the problems are in P for *every* fixed g and c: Thomassen proved that each fixed surface admits only finitely many 6-critical graphs [19] (for the torus, exactly four [18]), so $\chi(G) \le 5$ reduces to finitely many subgraph tests, and $\chi(G) \le r$ for $r \ge 6$ follows from degeneracy arguments; the crossing-number case reduces to the genus case using $g \le \mathrm{cr}$ together with [13], as worked out in [11]. Table 1 displays the situation.

	$r \le 3$	$r = 4$	$r \ge 5$
$g = 0$ (or $c = 0$)	NP-complete	P (trivial by 4CT)	P (trivial)
$g \ge 1$ fixed	NP-complete	**open**	P [19]
$c \ge 1$ fixed	NP-complete	**open**	P [11, 13]

Table 1: The complexity of $\{G : \chi(G) \le r \wedge g(G) \le g\}$ and $\{G : \chi(G) \le r \wedge \mathrm{cr}(G) \le c\}$.

The open row is thus exactly $r = 4$. The essential difficulty is that, unlike the situation for six colors, every surface other than the sphere carries infinitely many 5-critical graphs [8, 19], so no finite list of obstructions can settle 4-colorability, and the infinite families are not (yet) understood well enough to be tested for.

3 Progress: graphs with one crossing

The main development since [11] is a 2025 paper of Dvořák, Lidický, and Mohar [7] that attacks the smallest open cell, $\mathrm{cr}(G) \le 1$, head on. Let $\mathcal{C}$ denote the class of graphs drawable in the plane with at most one crossing; note $\mathcal{C}$ is contained in both the class of projective-planar and the class of toroidal graphs, so it is the natural warm-up for both $g = 1$ cases (orientable and not).

Their starting point is an inductive scheme modeled on the theory of 4-critical triangle-free graphs on surfaces developed by Dvořák, Král̆, and Thomas, which culminated in a linear-time algorithm for 3-colorability of triangle-free graphs on any fixed surface [6]. If $G \in \mathcal{C}$ is 5-critical and v is a vertex of degree four with nonadjacent neighbors x, y positioned away from the crossing, then deleting v and identifying x with y yields a smaller graph in $\mathcal{C}$ every 4-coloring of which lifts to G; criticality then produces a smaller 5-critical graph in $\mathcal{C}$, and one can hope to induct down to K_5. For the induction to have a base case one must handle 5-critical graphs of minimum degree five, and here [7] conjecture there are none:

Conjecture 1 (Dvořák–Lidický–Mohar [7])**.** *Every* 5*-critical graph in* $\mathcal{C}$ *contains a vertex of degree four. Equivalently (in their normalized setting), every graph of minimum degree five with no separating triangles drawn in the plane with one crossing is* 4*-colorable.*

They verify Conjecture 1 by computer for all graphs on at most 28 vertices, and prove, assuming it, that every non-4-colorable graph in their normalized subclass of $\mathcal{C}$ is obtained from K_5 by a sequence of explicit expansion operations—a rough structural characterization of the 5-critical graphs in $\mathcal{C}$. This is precisely the kind of theorem that has historically been converted into polynomial-time decision algorithms, as happened in the triangle-free program [6]. The authors are candid about the surfaces themselves: even for the projective plane and the torus, characterizing the 5-critical graphs, or merely obtaining a polynomial-time 4-colorability algorithm, "seems far beyond our reach using the current methods" [7].

4 Why hardness looks unlikely: crossings are expensive

Gasarch et al. observed [11] that NP-hardness proofs seem hard to come by because a gadget using one crossing cannot be reused. A simple identification argument, used as a normalization step in [7], turns this intuition into structure.

Lemma 2. *Let G be drawn in the plane, and let edges u_1v_1 and u_2v_2 cross. If $u_1u_2 \notin E(G)$, let G' be obtained from G by identifying u_1 and u_2. Then G' inherits a drawing with one fewer crossing, and every* 4*-coloring of G' induces a* 4*-coloring of G.*

Proof. Place the identified vertex at the crossing point, routing the edges formerly incident to u_1 along the segment of u_1v_1 from u_1 to the crossing, and similarly for u_2; the two crossing edges become the edges from the new vertex to v_1 and v_2. No new crossings are created and the chosen crossing disappears. A proper 4-coloring of G' pulls back by giving u_1 and u_2 the color of the identified vertex, which is proper since $u_1u_2 \notin E(G)$. □

Contrapositively, if G is not 4-colorable then neither is G'. Iterating and invoking the Four Color Theorem at the bottom yields:

Corollary 3. *Every non-*4*-colorable graph drawn in the plane with $c \geq 1$ crossings yields, after at most c identifications, a non-*4*-colorable graph drawn with at least one and at most c crossings in which the four endpoints of every crossing induce a K_4. In particular, in every one-crossing drawing of a non-*4*-colorable graph $G \in \mathcal{C}$, the four endpoints of the crossing induce a K_4.*

Informally, every crossing in a hard instance must be *clique-locked*: the nonplanarity it provides is confined inside a K_4 and carries only a bounded amount of information. All known NP-hardness reductions for coloring spend $\Theta(1)$ crossings per gadget occurrence and therefore produce instances with $\mathrm{cr} = \Theta(n)$; Corollary 3 suggests any reduction to bounded crossing number would have to encode an n-bit instance into $O(c)$ clique-locked crossings, which is hard to imagine. We note, for calibration, that a graph drawn with c crossings becomes planar after deleting at most c edges, hence has treewidth $O(\sqrt{n}) + c$, so 4-colorability is decidable in time $2^{O(\sqrt{n}+c)}$; as with planar 3-colorability, such subexponential algorithms are compatible with NP-completeness under the Exponential Time Hypothesis [12], but they do constrain the shape of any hypothetical reduction.

5 An experiment: reducibility with a crossing in the room

If Conjecture 1 is to be proved, the natural route is the one that proved the Four Color Theorem: exhibit an *unavoidable set* of configurations via a discharging argument, and show each configuration *reducible*, meaning it cannot occur in a minimal counterexample because any coloring of its boundary ring can be massaged, using Kempe chains, into one that extends inside. Dvořák, Lidický, and Mohar report a serious obstacle: many configurations that are D-reducible in the planar setting cease to be reducible in the presence of the crossing [7]. The following small experiment quantifies the phenomenon on the most famous configuration of all.

Recall the *Birkhoff diamond*: four internal vertices, inducing K_4 minus an edge and each of degree five in the ambient triangulation, surrounded by a ring C_6. Birkhoff proved it reducible in 1913 [3], launching the entire program. The ring C_6 has 732 proper 4-colorings, of which 384 extend directly to the interior. D-reducibility asks that every ring coloring be *good*, where the good set is the closure of the extendable colorings under the following adversarial Kempe step. Viewing the four colors as the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$, a Kempe swap of the color pair $\{a, b\}$ with $a \oplus b = t$ recolors, by translation by t, one connected component of the subgraph induced by a coset of $\{0, t\}$. In the exterior disk such a component meets the ring in a union of maximal arcs of same-coset vertices, and—this is where planarity enters—the grouping of arcs into components is a partition with coset-pure blocks that is *noncrossing* in the cyclic order, since disjoint connected sets in a disk attached to interleaved boundary arcs would have to cross. A ring coloring θ is good if for some t, however the adversary partitions the arcs subject to these constraints, some subset of blocks can be flipped to reach a coloring already known to be good.

We implemented this closure exactly.[2] Three adversaries were run. With the planar (noncrossing) adversary, all 732 ring colorings become good after three rounds: the Birkhoff diamond is D-reducible, as Birkhoff knew. With an adversary allowed *one simple crossing*—at most one pair of blocks may interleave, and only with four alternations in cyclic order, which is what two connected sets crossing once can produce—300 of the 348 non-extendable colorings can no longer be rescued. With chains permitted to cross freely, all 348 survive: not a single Kempe rescue remains.

The middle number is the interesting one. A single crossing available to the exterior does

[2] Python source available at [5]; the entire computation runs in seconds. The extendability count and the planar-case verdict serve as ground-truth checks against [3].

not merely weaken the Birkhoff diamond's reducibility; it destroys roughly 86% of the Kempe arguments the planar proof relies on. This is a toy model—the actual setting of [7] fixes the location of the crossing, works with precolored crossing endpoints (which Corollary 3 confines to a K_4), and has C-reducibility with contractions available, none of which we modeled—but it makes vivid why the discharging program for Conjecture 1 must be rebuilt rather than transplanted, and why the crossing must be quarantined before any reducibility can start.

6 Open problems

The original questions of [11] remain open and we restate the smallest one first.

Open Problem 1. *Is $\{G : \chi(G) \leq 4 \wedge \mathrm{cr}(G) \leq 1\}$ in P? Is it NP-complete? In particular, does Conjecture 1, or the expansion characterization of [7] that follows from it, yield a polynomial-time algorithm, as the analogous characterization did for triangle-free graphs on surfaces [6]?*

Open Problem 2. *Is 4-COLORING fixed-parameter tractable when parameterized by crossing number? By genus? Even membership in XP—a polynomial-time algorithm for each fixed c or g, which is exactly the question of [11]—is open; FPT is the uniform strengthening, and Corollary 3 suggests an induction on the number of crossings as a possible route for the crossing-number version.*

Open Problem 3. *Determine the slowest-growing function $c(n)$ for which 4-coloring graphs given with drawings having at most $c(n)$ crossings is NP-hard. Standard reductions give $c(n) = \Theta(n)$. Can hardness be pushed to $c(n) = n^{\varepsilon}$, or even $c(n) = \mathrm{polylog}(n)$? Alternatively, can one show, perhaps under ETH and using Corollary 3, that $o(n/\log n)$ clique-locked crossings cannot encode general instances?*

Open Problem 4. *Develop a reducibility theory for the one-crossing regime. Our experiment shows the Birkhoff diamond is not D-reducible against an adversary granted one simple chain crossing. Which configurations, if any, are? Does D- or C-reducibility with the four crossing endpoints precolored (and spanning a K_4) recover enough configurations to support a discharging proof of Conjecture 1?*

Open Problem 5. *On the torus, Thomassen's four 6-critical graphs [18] make $\chi(G) \leq 5$ testable in polynomial time, so the entire mystery of the toroidal $r = 4$ cell is distinguishing $\chi = 4$ from $\chi = 5$. Is there a polynomial-time algorithm for 4-colorability of toroidal graphs of sufficiently large edge-width (i.e., locally planar graphs)? Even this restricted question appears open.*

Our own view, for what it is worth: the evidence now points fairly clearly toward P for the entire $r = 4$ row, with the one-crossing case falling first—but only after someone re-runs a substantial portion of the Four Color Theorem's reducibility-plus-discharging machinery in a setting where, as the experiment above shows, most of the classical Kempe arguments are simply gone. The complexity-theoretic questions (Open Problems 2 and 3), by contrast, might be attackable without any discharging at all, and deserve attention from readers of this column in particular: this corner of the subject has so far been worked almost entirely by graph theorists.

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