

**Open Problems Column**  
**Edited by William Gasarch**

**This Issue's Column!** This issue's Open Problem Column is by Ben Weisz and is *The Rank of the Balanced Tripartitioning Tensor*

**Request for Columns!** I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere in between,

**The Rank of the Balanced Tripartitioning Tensor  $P_3$**

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**Abstract**

Since 2023 a line of work has tied the asymptotic rank of one family of tensors, the balanced tripartitioning tensors  $P_d$ , to two well-known questions: Strassen's asymptotic rank conjecture and the Set Cover Conjecture. Low rank for  $P_d$  would refute the Set Cover Conjecture and would give arithmetic circuits for the permanent that are exponentially smaller than Ryser's formula. A companion column treats  $P_2$ , a  $15 \times 15 \times 15$  tensor. Here we treat the next case,  $P_3$ , an  $84 \times 84 \times 84$  tensor. We survey the background, starting from what a tensor is, and state what is known:  $150 \leq \mathbf{R}(P_3) \leq \mathbf{R}(P_3) \leq 247$  over  $\mathbb{C}$ . The lower bound 150 is new and not yet peer reviewed (see the status note at the end); it replaces the flattening bound 84. The upper bound is due to Flavi, Jelisiejew and Michałek. We also explain why Pratt's group construction, with maps into  $\mathbb{Z}_2^m$ , cannot go below  $2^8 = 256$  for  $P_3$ , and pose problems about closing the gap.

## 1 Tensors and their rank

**Tensors.** Let  $\mathbb{F}$  be a field. For our purposes a (three-way) *tensor* is simply a three-dimensional array  $T = (T_{ijk})$  of elements of  $\mathbb{F}$ , with  $1 \leq i \leq n_1$ ,  $1 \leq j \leq n_2$ ,  $1 \leq k \leq n_3$ . The set of all such arrays is denoted  $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$ . For vectors  $a \in \mathbb{F}^{n_1}$ ,  $b \in \mathbb{F}^{n_2}$ ,  $c \in \mathbb{F}^{n_3}$ , the *outer product*  $a \otimes b \otimes c$  is the array with entries  $(a \otimes b \otimes c)_{ijk} = a_i b_j c_k$ .

**Example.** Let  $e_1, e_2$  be the standard basis of  $\mathbb{F}^2$ . Then  $e_1 \otimes e_2 \otimes e_1$  is the  $2 \times 2 \times 2$  array with a single 1, in position  $(1, 2, 1)$ , and  $(e_1 + e_2) \otimes (e_1 + e_2) \otimes (e_1 + e_2)$  is the  $2 \times 2 \times 2$  array of all ones. The array with  $T_{111} = T_{222} = 1$  and all other entries 0 equals  $e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$ .

**Trilinear forms.** Equivalently, a tensor  $T$  is the *trilinear form*

$$T(x, y, z) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} T_{ijk} x_i y_j z_k, \quad x \in \mathbb{F}^{n_1}, y \in \mathbb{F}^{n_2}, z \in \mathbb{F}^{n_3},$$

a polynomial that is linear in each of the three vectors of variables separately. The outer product  $a \otimes b \otimes c$  corresponds to the product of three linear forms,  $(\sum_i a_i x_i)(\sum_j b_j y_j)(\sum_k c_k z_k)$ . Everything generalises to  $m$ -way arrays and  $m$ -linear forms (for  $m = 4$ ,  $\sum T_{ijkl} x_i y_j z_k w_l$ ); here we mainly need  $m = 3$ , the case that governs bilinear maps such as matrix multiplication.

**Example** (matrix multiplication). The matrix multiplication tensor  $\langle n, n, n \rangle$  has its three index sets indexed by pairs, and corresponds to the trilinear form

$$\langle n, n, n \rangle(X, Y, Z) = \sum_{i,j,k=1}^n X_{ij} Y_{jk} Z_{ki} = \text{trace}(XYZ).$$

The coefficient of  $Z_{ki}$  is  $\sum_j X_{ij} Y_{jk} = (XY)_{ik}$ , so this form encodes the product  $XY$ .

**Rank.** A tensor has *rank one* if it is of the form  $a \otimes b \otimes c$  with nonzero vectors. The *rank*  $R(T)$  is the smallest  $r$  such that  $T$  is a sum of  $r$  rank-one tensors. Over  $\mathbb{C}$ , the *border rank*  $\underline{R}(T)$  is the smallest  $r$  such that  $T$  is a limit of tensors of rank at most  $r$ ; clearly  $\underline{R}(T) \leq R(T)$ .

**Rank counts multiplications.** A tensor  $T$  also defines a *bilinear map*  $B_T: \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \rightarrow \mathbb{F}^{n_3}$ , whose  $k$ -th output is the coefficient of  $z_k$  in  $T(x, y, z)$ :  $B_T(x, y)_k = \sum_{i,j} T_{ijk} x_i y_j$ . A *bilinear algorithm* for  $B_T$  computes some products  $m_1, \dots, m_r$ , each of the form  $m_t = (\text{linear form in } x) \cdot (\text{linear form in } y)$ , and outputs linear combinations of  $m_1, \dots, m_r$ . Additions and multiplications by constants are free; only the  $r$  multiplications  $m_t$  are counted.

**Proposition 1.** *The smallest number of multiplications in a bilinear algorithm for  $B_T$  equals  $R(T)$ .*

*Proof.* If  $T = \sum_{t=1}^r a_t \otimes b_t \otimes c_t$ , compute  $m_t = (\sum_i a_{t,i} x_i)(\sum_j b_{t,j} y_j)$  for  $t = 1, \dots, r$  and output  $B_T(x, y)_k = \sum_t c_{t,k} m_t$ . Conversely, if an algorithm computes  $m_t = (a_t \cdot x)(b_t \cdot y)$  and outputs  $B_T(x, y)_k = \sum_t c_{t,k} m_t$  for all  $x, y$ , then comparing coefficients of  $x_i y_j$  gives  $T_{ijk} = \sum_t a_{t,i} b_{t,j} c_{t,k}$ , that is,  $T = \sum_t a_t \otimes b_t \otimes c_t$ .  $\square$

**Example** (complex multiplication). Multiplying  $x_1 + x_2 i$  by  $y_1 + y_2 i$  over  $\mathbb{R}$  means computing  $x_1 y_1 - x_2 y_2$  and  $x_1 y_2 + x_2 y_1$ . The obvious method uses 4 multiplications. Gauss's trick uses 3:  $m_1 = x_1 y_1$ ,  $m_2 = x_2 y_2$ ,  $m_3 = (x_1 + x_2)(y_1 + y_2)$ , with outputs  $m_1 - m_2$  and  $m_3 - m_1 - m_2$ . So the corresponding  $2 \times 2 \times 2$  tensor has rank at most 3 over  $\mathbb{R}$ .

**Example** (Strassen). Strassen's algorithm [Str69] multiplies  $2 \times 2$  matrices with 7 multiplications instead of 8, so  $R(\langle 2, 2, 2 \rangle) \leq 7$ . Seven is optimal:  $R(\langle 2, 2, 2 \rangle) = 7$  [HK71, Win71], and even  $\underline{R}(\langle 2, 2, 2 \rangle) = 7$  [Lan06]. Applied recursively to blocks, it multiplies  $n \times n$  matrices in  $O(n^{\log_2 7})$  operations.

Since then the exponent of matrix multiplication has been lowered many times. The best bound published in a peer-reviewed venue that we know of is  $\omega < 2.371339$  [ADV<sup>+</sup>25]; a 2026 preprint lowers it to  $\omega < 2.371177$  [DEK<sup>+</sup>26], and an OpenAI preprint of October 2026 claims  $\omega \leq 9/4$  over  $\mathbb{C}$  [OAI26], which has not yet been peer reviewed. None of this will be our concern.

**Kronecker products and asymptotic rank.** The same symbol is used for the *Kronecker product* of two tensors: for  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  and  $T' \in \mathbb{F}^{n'_1} \otimes \mathbb{F}^{n'_2} \otimes \mathbb{F}^{n'_3}$ , the tensor  $T \otimes T'$  has entries  $(T \otimes T')_{(i,i'),(j,j'),(k,k')} = T_{ijk} T'_{i'j'k'}$ . Rank is submultiplicative,  $R(T \otimes T') \leq R(T) R(T')$ ; for example  $\langle n, n, n \rangle \otimes \langle m, m, m \rangle = \langle nm, nm, nm \rangle$ , which is the recursion behind Strassen's algorithm. The *asymptotic rank* of  $T$  is

$$\tilde{R}(T) = \lim_{k \rightarrow \infty} R(T^{\otimes k})^{1/k}.$$

The limit exists and equals the infimum, by submultiplicativity. Replacing rank by border rank gives the same value; see [BCS97, Lemma 15.27], as cited in [Pra24]. For matrix multiplication,  $\tilde{R}(\langle n, n, n \rangle) = n^\omega$ , where  $\omega$  is the exponent of matrix multiplication.

**Concise and tight tensors.** Two structural properties of the array  $T$  are defined below. We follow [Pra23, Pra24].

**Definition.** A tensor  $T$  is *concise* if it cannot be written, after a change of basis in one of the three factors, with fewer variables in that factor; equivalently, its three flattenings (the  $n_1 \times n_2 n_3$  matrix with entries  $T_{i,(j,k)}$ , and the two analogous ones) have ranks  $n_1$ ,  $n_2$  and  $n_3$ . A tensor  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  is *tight* if, for some choice of bases, there are injective maps  $f, g, h: [n] \rightarrow \mathbb{Z}$  with  $f(i) + g(j) + h(k) = 0$  for every  $(i, j, k)$  with  $T_{ijk} \neq 0$ .

Conciseness says no coordinate is wasted. Tightness says the support of  $T$  lies on a “plane” after relabelling, as for matrix multiplication. If  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  is concise, a flattening has rank  $n$ , so  $R(T) \geq \underline{R}(T) \geq n$  and  $\tilde{R}(T) \geq n$ .

## 2 Strassen’s asymptotic rank conjecture

In the 1980s Strassen developed a theory of the asymptotic behaviour of tensors, the asymptotic spectrum [Str88]. In [Str94] he formulated the following conjecture, stated here as in [Pra23, Pra24].

**Conjecture** (Asymptotic Rank Conjecture). *For every tight and concise tensor  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ ,  $\tilde{R}(T) = n$ .*

So the conjecture says that, asymptotically, tight concise tensors are as cheap as the lower bound  $n$  from conciseness allows. The matrix multiplication tensor  $\langle m, m, m \rangle$  is tight and concise with  $n = m^2$ . Hence the conjecture would imply  $\tilde{R}(\langle m, m, m \rangle) = m^2$ , that is,  $\omega = 2$  [Pra23]. Björklund, Kaski, Koana and Nederlof call it “a bold conjecture that has recently seen intriguing progress” [BKKN26].

**How the pieces fit together.** Two implications are known, and one hypothesis is widely believed:

- the Asymptotic Rank Conjecture implies  $\omega = 2$ ;
- the Asymptotic Rank Conjecture implies that the Set Cover Conjecture (Section 3) fails [BK24, Pra23]: the two conjectures are not both true;
- the Set Cover Conjecture, which says that for large enough fixed set size set cover essentially needs  $2^n$  time, is believed by many researchers.

So anyone who believes the Set Cover Conjecture must believe that the Asymptotic Rank Conjecture is false. This does *not* force  $\omega > 2$ :  $\omega = 2$  is a consequence of the Asymptotic Rank Conjecture, and the converse is not known. The known proofs that the two conjectures are incompatible apply the conjecture to specific tight tensors (in [Pra23], the tensors  $P_d$  below) rather than using  $\omega = 2$ , and Pratt notes that his results do not make  $\omega = 2$  inconsistent with the Set Cover Conjecture [Pra23, comment 3 after Corollary 1.12]. To our knowledge it is open whether  $\omega = 2$  alone contradicts the Set Cover Conjecture (Open Problem 6).

## 3 The Set Cover Conjecture

In the  $s$ -Set Cover problem we are given a universe  $[n]$ , a family  $\mathcal{F}$  of subsets of  $[n]$ , each of size at most  $s$ , and an integer  $t$ . We must decide whether at most  $t$  sets in  $\mathcal{F}$  cover  $[n]$ . Dynamic programming over the subsets of the universe solves it in time  $2^n \cdot \text{poly}(n, |\mathcal{F}|)$ . The Set Cover Conjecture, formulated by Cygan et al. [CDL<sup>+</sup>16] (see also [KT19]), says that this is essentially optimal even for bounded set size. Here it is stated as in [Pra23, Pra24].

**Conjecture** (Set Cover Conjecture). *For every  $\varepsilon > 0$  there exists  $s \in \mathbb{N}$  such that no (possibly randomized) algorithm solves  $s$ -Set Cover in time  $2^{(1-\varepsilon)n}$ .*

## 4 From tensors to set cover

Björklund and Kaski [BK24] showed that the Asymptotic Rank Conjecture and the Set Cover Conjecture are not both true. Pratt [Pra23, Pra24] gave a short proof and a sharper statement. It goes through the *balanced tripartitioning* problem: given families  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$  of  $n$ -subsets of  $[3n]$ , decide whether there are  $S_i \in \mathcal{F}_i$  with  $S_1 \cup S_2 \cup S_3 = [3n]$ . It can be solved in time  $8^n \cdot \text{poly}(n)$  by Fourier analysis over  $\mathbb{Z}_2^{3n}$  [Pra23]. The key object is one family of tensors.

**Definition.** For  $d \geq 1$ , the *balanced tripartitioning tensor*  $P_d$  is the array indexed by triples  $(A, B, C)$  of  $d$ -element subsets of  $[3d]$ , with entry 1 if  $A, B, C$  are pairwise disjoint (so that  $A \cup B \cup C = [3d]$ ) and 0 otherwise. As a trilinear form,

$$P_d(x, y, z) = \sum_{\substack{A, B, C \subseteq [3d], \\ |A|=|B|=|C|=d \\ A \cup B \cup C = [3d]}} x_A y_B z_C,$$

a tensor in  $\mathbb{F}^N \otimes \mathbb{F}^N \otimes \mathbb{F}^N$  with  $N = \binom{3d}{d}$ . Pratt denotes it  $T_d$  [Pra23, Pra24]; Björklund, Kaski, Koana and Nederlof denote it  $P_d$  [BKKN26].

Evaluating  $P_d$  on indicator vectors counts solutions: if  $\chi_i$  is the indicator vector of  $\mathcal{F}_i$ , then  $P_d(\chi_1, \chi_2, \chi_3)$  is the number of triples  $(S_1, S_2, S_3) \in \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3$  that partition  $[3d]$ . By Proposition 1, a rank- $r$  decomposition of  $P_d$  computes all  $N$  values  $B_{P_d}(\chi_1, \chi_2)_C$  with  $r$  multiplications, and the count is the sum of those with  $C \in \mathcal{F}_3$ .

$P_d$  is concise and tight [Pra23, Proposition 2.1]; since  $P_d$  is an array, both definitions apply to it directly. The flattening of  $P_d$  along the first factor has, in row  $A$ , ones exactly in the columns  $(B, C)$  with  $B \cup C = [3d] \setminus A$ ; distinct rows have disjoint supports, so it has rank  $N$ , and by the symmetry of  $P_d$  in its three factors the other two flattenings behave the same way. For tightness one can take  $f(A) = \sum_{a \in A} 2^a$ ,  $g(B) = \sum_{b \in B} 2^b$  and  $h(C) = \sum_{c \in C} 2^c - \sum_{m=1}^{3d} 2^m$ ; these are injective, and  $f(A) + g(B) + h(C) = 0$  whenever  $A, B, C$  partition  $[3d]$ . So the Asymptotic Rank Conjecture predicts  $\tilde{R}(P_d) = \binom{3d}{d}$ .

**Theorem 1** (Pratt [Pra23, Pra24]).

1. *For every  $k$  and  $\varepsilon > 0$ , balanced tripartitioning has a randomized algorithm with running time  $((\tilde{R}(P_k) + \varepsilon) \cdot 27^k / (\binom{3k}{k} \binom{2k}{k}))^{n/k}$ .*
2. *Consequently, the Asymptotic Rank Conjecture implies a randomized algorithm for  $s$ -Set Cover with running time  $(3/2^{2/3} + \varepsilon)^n$ , for every  $\varepsilon > 0$  and every fixed  $s$ .*
3. *In the other direction, the Set Cover Conjecture implies  $\tilde{R}(P_k) > (8 - \varepsilon)^k$  for every  $\varepsilon > 0$  and all large enough  $k$ . It also implies*

$$R(P_k) \geq \theta_k := \frac{8^k \binom{3k}{k} \binom{2k}{k}}{27^k} \quad \text{for every } k.$$

*Proof idea* (see [Pra23] for the details). Split  $[3n]$  at random into  $n/k$  blocks of  $3k$  elements. A fixed solution  $(S_1, S_2, S_3)$  is *compatible* with the split if every block meets each  $S_i$  in exactly  $k$  elements;

this happens with probability about  $((\binom{3k}{k}\binom{2k}{k})/27^k)^{n/k}$ , up to polynomial factors. Compatible solutions correspond to the monomials of the Kronecker power  $P_k^{\otimes n/k}$ . Zeroing the variables of sets outside the three families leaves a polynomial that is nonzero exactly when a compatible solution exists; Pratt evaluates it at random points (Schwartz–Zippel lemma), and in characteristic 0 one may instead evaluate it at the indicator vectors, which counts the compatible solutions. A Kronecker-structured decomposition of  $P_k^{\otimes n/k}$  with about  $(\tilde{R}(P_k) + \varepsilon)^{n/k}$  terms evaluates it in roughly that many steps. Repeating for about  $(27^k / ((\binom{3k}{k}\binom{2k}{k})))^{n/k}$  random splits gives part 1. Part 2 combines part 1 with a reduction from set cover to balanced tripartitioning; part 3 is the contrapositive of the same combination, used with the actual value of  $\tilde{R}(P_k)$  (or  $R(P_k)$ ) instead of the value the conjecture predicts.  $\square$

So a single good rank bound, for a single  $k$ , would refute the Set Cover Conjecture. The next proposition shows that this cannot happen for small  $k$ .

**Proposition 2** (essentially [Pra23]; written proof in [W26a]). *For every field and every  $k \geq 1$ ,  $R(P_k) \geq \binom{3k}{k}$ . Moreover  $\theta_k < \binom{3k}{k}$  for  $1 \leq k \leq 10$  and  $\theta_k > \binom{3k}{k}$  for  $k \geq 11$ . Hence no upper bound on  $R(P_k)$  with  $k \leq 10$  can violate the inequality  $R(P_k) \geq \theta_k$  of part 3 of Theorem 1.*

*Proof.* The first claim is conciseness: the flattening above has rank  $\binom{3k}{k}$ , and a sum of  $r$  rank-one tensors has flattenings of rank at most  $r$ . For the second, put  $\rho_k = \theta_k / \binom{3k}{k} = 8^k \binom{2k}{k} / 27^k$ . Then  $\rho_{k+1} / \rho_k = 16(2k+1) / (27(k+1))$ , which exceeds 1 exactly when  $k \geq 3$ . Direct computation gives  $\rho_k < 1$  for  $k \leq 10$  (for example  $\rho_{10} \approx 0.964$ ) and  $\rho_{11} \approx 1.090 > 1$ , and the monotonicity for  $k \geq 3$  extends this to all  $k$ .  $\square$

The second statement is the same comparison,  $8^k \binom{2k}{k}$  versus  $27^k$ , as Pratt’s remark that, assuming the Asymptotic Rank Conjecture,  $k = 11$  is the smallest value for which the algorithm of Theorem 1 would beat the  $8^n \cdot \text{poly}(n)$  time bound [Pra23].

Björklund, Kaski, Koana and Nederlof [BKKN26] showed that sufficiently low tensor rank of  $P_d$ , for a large enough constant  $d$ , implies uniform arithmetic circuits for the matrix permanent that are exponentially smaller than those obtainable from Ryser’s formula. It also implies exponential-time improvements for a wide variety of counting and decision problems. Their key tool is a “Kronecker scaling” property of the tensors  $P_n$ . They also show that the low-rank assumption follows from the Asymptotic Rank Conjecture.

In short, the central question is how fast  $R(P_d)$  grows with  $d$ .

## 5 What is known for $P_3$

**The tensor.**  $P_3$  is indexed in each factor by the  $N = \binom{9}{3} = 84$  three-element subsets of  $[9]$ , and it has  $\binom{9}{3}\binom{6}{3} = 1680$  nonzero coefficients, all equal to 1. It is concise and tight (see the discussion before Theorem 1), so the Asymptotic Rank Conjecture predicts  $\tilde{R}(P_3) = 84$ . The Set Cover Conjecture says nothing about this single size: by Proposition 2,  $\theta_3 = 8^3 \binom{9}{3}\binom{6}{3} / 27^3 = 860160 / 19683 \approx 43.7$  is far below the trivial lower bound 84. For the permanent, what matters in [BKKN26] is the exponent of the whole sequence, the infimum of the  $\sigma$  with  $R(P_d) \leq \binom{3d}{d}^{\sigma + o(1)}$ . Any value below  $1/H(1/3) \approx 1.089$ , the value given by Pratt’s bound  $R(P_d) \leq 2^{3d-1}$ , would give the improvement over Ryser’s formula [Rys63]; here  $H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$  is the binary entropy function. A single  $d$  is not comparable with this threshold: at  $d = 3$  the known bounds give  $\log R(P_3) / \log 84$  between  $\log 150 / \log 84 \approx 1.131$  and  $\log 247 / \log 84 \approx 1.243$ , and Pratt’s bound

256 gives  $\approx 1.252$ . As for  $P_2$ , the point of  $P_3$  is that it is a concrete case on which constructions and lower-bound methods can be compared.

**Smaller cases, for comparison.**  $R(P_1) = \underline{R}(P_1) = 4$  in characteristic 0, where  $P_1$  is the  $3 \times 3$  permanent viewed as a trilinear form. Glynn’s formula [Gly10] gives the upper bound, Ilten and Teitler [IT16] the rank lower bound, and Derksen and Makam the border-rank lower bound (see [HS26]). For  $P_2$ ,  $26 \leq \underline{R}(P_2) \leq R(P_2) \leq 29$  in characteristic 0 [W26a, W26b].

### Upper bounds.

- $R(P_d) \leq 2^{3d-1}$  over every field of characteristic  $\neq 2$ , by Pratt’s group construction [Pra23, arXiv v1, item 4 after Corollary 1.12]. Under one reading the printed map is valid exactly for odd  $d$ , so for  $d = 3$ ; a corrected map gives the bound for every  $d$  [W26c]. For  $d = 3$  this gives 256.
- $R(P_d) \leq \sum_{j=0}^d \binom{3d}{2j}$  over algebraically closed fields of characteristic 0 [FJM25, arXiv v2, Proposition 6.18], and the same bound holds over every field of characteristic  $\neq 2$ , with an explicit integer identity [W26e, Theorem 2 and Corollary 1]. For  $d = 3$  this gives  $1+36+126+84 = 247$ .

### Lower bounds.

- $\underline{R}(P_3) \geq 84$ , the flattening bound (conciseness; see [BKKN26]). This was the best lower bound we found in the literature.
- $\underline{R}(P_3) \geq 150$  over  $\mathbb{C}$ , and over every field of characteristic 0 or of prime characteristic at most 31 [W26d]. (Over fields other than  $\mathbb{C}$ , border rank is defined algebraically, by degenerations; see [W26d].) The proof is a Koszul–Young flattening [LO15] with  $p = 4$ . Concretely, an explicit linear map  $\Phi: \mathbb{C}^{84} \rightarrow \mathbb{C}^9$  is applied to the first factor, which can only lower border rank. The projected tensor  $T'$  defines a linear map  $\Lambda^4 \mathbb{C}^9 \otimes (\mathbb{C}^{84})^* \rightarrow \Lambda^5 \mathbb{C}^9 \otimes \mathbb{C}^{84}$ , by wedging with the first factor of  $T'$  and contracting with the second; its matrix has size  $126 \cdot 84 = 10\,584$ . This integer matrix has rank 10 500 modulo every odd prime  $q \leq 31$ , hence at least 10 500 over  $\mathbb{Q}$ , and 10 498 modulo 2. A rank-one tensor gives rank  $\binom{8}{4} = 70$ , and  $10\,498 > 149 \cdot 70 = 10\,430$ , so  $\underline{R}(P_3) \geq 150$ . Koszul–Young flattenings are the tool behind the lower bound  $2n^2 - n$  for the border rank of  $\langle n, n, n \rangle$  [LO15].

Together, over  $\mathbb{C}$ ,

$$150 \leq \underline{R}(P_3) \leq R(P_3) \leq 247.$$

**What the lower bound means for algorithms.** By Proposition 1, over  $\mathbb{C}$  and over the other fields covered by [W26d], every bilinear algorithm for the map  $B_{P_3}$ , from whose outputs the solutions of a size-3 balanced tripartitioning instance are counted, needs at least 150 multiplications. Since the bound is on the border rank, the same holds over  $\mathbb{C}$  for approximate bilinear algorithms in the sense of Bini et al., whose least number of multiplications is the border rank (see [BCS97]).

**The limit of the method.** A *linear flattening* is a linear map  $L$  from tensors to matrices. If  $L$  maps every rank-one tensor to a matrix of rank at most  $\rho$ , then  $\underline{R}(T) \geq \text{rank } L(T)/\rho$ ; the bound above is of this kind, with  $\rho = 70$ . Over  $\mathbb{C}$ , Lovitz (preprint) [Lov26, Corollary 1.3] proved that this method cannot certify a border rank lower bound above  $2n - 1$  for an  $n \times n \times n$  tensor, improving the earlier barrier  $8n$  of [EGOW18] and later ones (see [Lov26]). For  $P_3$  this barrier is 167: flattenings

could in principle certify a little more than 150, but never the upper bound 247. Methods beyond linear flattenings do exist: Lovitz [Lov26, Theorem 1.1] also constructs explicit  $n \times n \times n$  tensors of border rank at least  $3n - o(n)$ .

## 6 Group constructions over $\mathbb{Z}_2^m$ cannot go below 256 for $P_3$

Pratt's upper bound comes from a simple principle. Let  $G$  be a finite abelian group, let  $f_1, f_2, f_3$  be maps from the  $d$ -subsets of  $[3d]$  to  $G$ , and let  $x \in G$ . Call  $(f_1, f_2, f_3, x)$  *valid* if  $f_1(S) + f_2(T) + f_3(U) = x$  holds exactly when  $(S, T, U)$  is an ordered partition of  $[3d]$ . Zeroing out and relabelling variables then turns the group tensor  $\sum_{a+b+c=x} e_a \otimes e_b \otimes e_c$  of  $G$  into  $P_d$ , so  $R(P_d)$  is at most the rank of that tensor. This rank is  $|G|$  over  $\mathbb{C}$  for every  $G$ , by the characters of  $G$ , and over every field of characteristic  $\neq 2$  for  $G = \mathbb{Z}_2^m$ , by Fourier analysis on  $\mathbb{Z}_2^m$  [W26g, Theorem 1]. Pratt states the principle for a single map  $f_1 = f_2 = f_3$  [Pra23, arXiv v1, item 4 after Corollary 1.12]. The construction in [Pra23, W26c] uses  $G = \mathbb{Z}_2^{3d-1}$ . How small can  $G$  be? The following is shown in [W26g] (not yet peer reviewed).

- *Affine maps.* If each  $f_i(S) = L_i(\mathbf{1}_S) + b_i$  is affine over  $\mathbb{F}_2$ , the quadruple is valid if and only if  $L_1, L_2, L_3$  agree on the even-weight code  $E \subset \mathbb{F}_2^{3d}$ ,  $L_1$  is injective on  $E$ , and  $x$  has the value forced by one ordered partition. The key fact is that the vectors  $\mathbf{1} + \mathbf{1}_S + \mathbf{1}_T + \mathbf{1}_U$ , over all triples of  $d$ -subsets, are exactly  $E$ . Hence  $m \geq 3d - 1$ : affine constructions never beat  $2^{3d-1}$ , which is 256 for  $d = 3$ .
- *Arbitrary maps.* For arbitrary maps into any finite abelian group  $G$ , a valid quadruple forces each  $f_i$  to be injective, and  $|G| \geq 2 \binom{3d}{d} - \binom{2d}{d}$ . For  $d = 3$  this is  $168 - 20 = 148$ . For  $G = \mathbb{Z}_2^m$  it forces  $2^m \geq 148$ , that is,  $m \geq 8$ .

So for  $P_3$  no group construction over  $\mathbb{Z}_2^m$ , affine or not, goes below 256. The bound 247 of [FJM25] comes from a different construction, which keeps the characters of the group in only two of the three factors [W26e]. Groups of other types, of order between 148 and 246, are not excluded by these arguments.

## 7 Open problems

**Open Problem 1.** Narrow the gap  $150 \leq \underline{R}(P_3) \leq R(P_3) \leq 247$ .

The gap is much wider than for  $P_2$  (26 to 29). The two known upper bounds are close relatives. Pratt's group construction (256) restricts the group tensor of  $\mathbb{Z}_2^{3d-1}$ . The construction of [FJM25, arXiv v2, Proposition 6.18] (247) starts from the group tensor of  $\mathbb{Z}_2^{3d}$ , keeps its characters in two of the three factors and substitutes in the third; for  $P_d$  it improves  $2^{3d-1}$  only by a factor  $1 - O((27/32)^d)$  [W26e, Proposition 4]. On the other side are the flattening lower bounds.

**Open Problem 2.** Can some linear flattening, Koszul or other, certify  $\underline{R}(P_3) > 150$ ? Lovitz's barrier allows up to 167. Can a method beyond linear flattenings, such as those behind the explicit tensors of [Lov26], give a lower bound above 167 for  $P_3$ ?

**Open Problem 3.** Is there a finite abelian group  $G$  with  $148 \leq |G| < 247$  and maps  $f_1, f_2, f_3$  forming a valid quadruple for  $d = 3$ ? By Section 6,  $G$  cannot be of the form  $\mathbb{Z}_2^m$ .

**Open Problem 4.** Is  $\underline{R}(P_3) < R(P_3)$ ?

For  $\langle 2, 2, 2 \rangle$  rank and border rank coincide. Asymptotic rank does not distinguish the two, so a gap for  $P_3$  would matter for constructions but not for the conjectures themselves.

**Open Problem 5** (the motivating question, from [Pra23, BKKN26]). Is there an  $\varepsilon > 0$  with  $\tilde{R}(P_d) \leq (8 - \varepsilon)^d$  for all large  $d$ ?

A positive answer would refute the Set Cover Conjecture [Pra23, Pra24]. The Asymptotic Rank Conjecture predicts  $\tilde{R}(P_d) = \binom{3d}{d} \approx (27/4)^d$  up to polynomial factors.

**Open Problem 6.** Does  $\omega = 2$  alone imply that the Set Cover Conjecture fails? In algorithmic terms: would matrix multiplication in time  $n^{2+o(1)}$  yield an algorithm for  $s$ -Set Cover in time  $(2 - \varepsilon)^n$ , for some  $\varepsilon > 0$  and every fixed  $s$ ?

Consequences of  $\omega = 2$  for exponential-time algorithms do exist: Nederlof [Ned20] showed that quadratic-time matrix multiplication would solve the traveling salesman problem on bipartite graphs in time  $o(1.9999^n)$ .

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