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THE DECOMPOSITION OF A SQUARE INTO RECTANGLES OF MINIMAL PERIMETER

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This paper solves the problem of subdividing a unit square into p rectangles of area $1/p$ in such a way that the maximal perimeter of a rectangle is as small as possible. The correctness of the solution is proved using the well-known theorems of Menger and Dilworth.

Square decomposition

In this work we consider the following geometric decomposition problem.

Square decomposition. Given a unit square D and a positive integer p, subdivide D into p rectangles of area $1/p$ (all having edges parallel to those of D) in such a way that the maximum of their perimeters is minimized.

The square decomposition problem has applications in a number of areas such as bin packing of flexible objects of fixed area and circuit layout with minimum communication requirements. Our interest in the square decomposition problem arises from the following problem in parallel computation [1]. One wishes to compute a table of all values of a binary function f on the Cartesian product $S \times T$, where $|S| = |T| = m$. The computation is to be performed in parallel on p processing units. The function values are computed as follows. The i-th processor computes the values of f on some subset W_i of $S \times T$. The sets W_i, $1 \le i \le p$, partition $S \times T$. To minimize computation time, each processor is assigned an approximately equal number of function values to compute. Each processor has a small amount of local memory used for storing its operands. The objective is to minimize this storage. The

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amount of storage used by the i-th processor is equal to the sum of the projections of W_i onto the coordinate axes. When $p \ll m^2$ a good heuristic for this problem consists of solving the square decomposition problem (on a square of size m by m) and then approximating this decomposition on an $m \times m$ integer lattice.

We describe a simple procedure for the square decomposition problem and prove its correctness by giving a bound on the length of the longest and shortest sides of any rectangle in such a decomposition. Alon and Kleitman [2] consider a similar problem where there is no constraint on the areas of the rectangles, however they establish the exactness of their bound only when $p = n(n+1)$ or $p = n^2$ for some integer n.

The perimeter of a rectangle of given area A (in our case $A = 1/p$) is a strictly increasing (strictly decreasing) function of the length of its longest (shortest) side, because $(\sqrt{A}+h)+A/(\sqrt{A}+h)$ is a strictly increasing function of $h \geq 0$. One immediate consequence is that if p is the square of a positive integer n then the square decomposition problem is solved by subdividing D into squares of side-length $1/n$ in the obvious way.

Now suppose $n^2 < p < (n+1)^2$ for some positive integer n. If $p \leq n(n+1)$, then let $r = n(n+1) - p$ and let $s = p - n^2$; if $p > n(n+1)$, then let $r = (n+1)^2 - p$ and let $s = p - n(n+1)$. Then r and s are nonnegative and $rn + s(n+1) = p$. We claim that the square decomposition problem can be solved by subdividing D into r rows each consisting of n rectangles with side-lengths $(1/n, n/p)$, followed by s rows each consisting of $n+1$ rectangles with side-lengths $(1/(n+1), (n+1)/p)$ (see Fig. 1).

Plainly, $1/n$ is the longer side of the $(1/n, n/p)$ rectangles, and $1/(n+1)$ is the shorter side of the $(1/(n+1), (n+1)/p)$ rectangles. So, to prove that our suggested decomposition does indeed minimize the maximal perimeter, it suffices to show that any subdivision of D into rectangles must include a rectangle whose longest side has length at least $1/n$, and a rectangle whose shortest side has length at most $1/(n+1)$. Our proof of this makes essential use of the well-known graph theoretic results of Dilworth and Menger.

Theorem 1 (Dilworth). *If $(S, <)$ is a partially-ordered set and m is any positive integer, then either S is a union of m chains or S contains a subset of $m+1$ elements no two of which are comparable.*

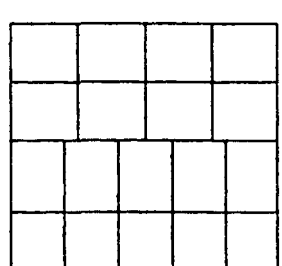

Fig. 1. Decomposition into $p = 18$ regions $(n = 4, r = s = 2)$.

Theorem 2 (Menger). *If s and t are distinct vertices of a graph G and m is any positive integer, then either there exist m paths from s to t no two of which have a vertex in common (other than s and t) or there exists a set S of $m-1$ vertices (not containing s or t) such that every path from s to t passes through a vertex in S.*

Theorem 2 is in fact equivalent to the celebrated 'Max-flow, Min-cut Theorem'. See [3] for a proof of this equivalence and direct proofs of Theorems 1 and 2. As we observed above, the following propositions imply the correctness of our solution to the square decomposition problem.

Proposition 1. *Let D be the unit square $\{(x,y)\,|\,0\leq x\leq 1, 0\leq y\leq 1\}$, and let n and p be positive integers such that $n^2<p$. Let $\{R_i\,|\,1\leq i\leq p\}$ be a collection of closed rectangles contained in D whose sides are parallel to the axes and whose interiors are pairwise disjoint. Then there is some rectangle R_i whose shortest side has length at most $1/(n+1)$.*

Proof. For $1\leq i\leq p$ let P_i denote the projection of R_i on the y-axis. Define a strict partial order $\lhd$ on $\{R_i\,|\,1\leq i\leq p\}$ by $R_i \lhd R_j$ if and only if $u\leq v$ for all u in P_i and v in P_j. By Theorem 1, either $\{R_i\,|\,1\leq i\leq p\}$ is a union of n chains or there exists a set B of $n+1$ rectangles R_i no two of which are comparable. In the first case one of the n chains, C say, contains at least $n+1$ rectangles, since $n^2<p$. The projections of any two rectangles in a chain are closed intervals on the y-axis over $[0,1]$ that are disjoint except possibly at their endpoints. So since the chain C contains at least $n+1$ rectangles there must be a rectangle in C of height at most $1/(n+1)$. In the second case the projections of any two rectangles in B on the x-axis must be closed intervals over $[0,1]$ that are disjoint except possibly at their endpoints. So one of the $n+1$ rectangles in B has width at most $1/(n+1)$. $\square$

Proposition 2. *Let D be the unit square $\{(x,y)\,|\,0\leq x\leq 1, 0\leq y\leq 1\}$, and let n and p be positive integers such that $p<(n+1)^2$. Let $\{R_i\,|\,1\leq i\leq p\}$ be a collection of closed rectangles whose sides are parallel to the axes and whose union is D. Then there is some rectangle R_i whose longest side has length at least $1/n$.*

Proof. We associate a graph G with $\{R_i\,|\,1\leq i\leq p\}$ so that there is a 1–1 correspondence between the vertices of G and the rectangles R_i, and two vertices of G are joined by an edge of G if and only if the boundaries of the rectangles corresponding to the vertices meet each other. Let e_1 and e_2 be the sides of D that lie on the line $x=0$ and the line $x=1$ respectively. Add two vertices s and t to G, join s to all vertices in G that correspond to rectangles whose boundry meets e_1, and join t to all vertices in G that correspond to rectangles whose boundary meets e_2. Call the resulting graph G'.

By Theorem 2, either there exist $n+1$ paths in G' from s to t no two of which have a vertex in common (other than s and t), or there exists a set S of n vertices in G such that every path in G' from s to t passes through some vertex in S. In the first case one of the $n+1$ paths, P say, contains no more than n vertices in G, since $(n+1)^2 > p$. A path in G' from s to t corresponds to a sequence of contiguous rectangles, in which the first member is in contact with e_1 and the last member is in contact with e_2. Since there are at most n rectangles in the sequence corresponding to P one of the rectangles in the sequence has width at least $1/n$. In the second case the n rectangles corresponding to the vertices in S must separate e_1 from e_2, so at least one of them has height $1/n$ or greater. $\square$

There are natural ways of generalizing the square decomposition problem:

(i) Replace D by an arbitrary rectangle.

(ii) Consider subdivisions of a cube into p cuboids of volume $1/p$ with the objective of making the maximum of the surface-areas of the cuboids as small as possible; generalizations to higher dimensions are obvious.

(iii) Consider a collection of measurable sets $\{S_i \mid 1 \le i \le p\}$ whose union is a square (or rectangle); the objective is to make the maximum (over i) of the sum of the horizontal and vertical projections of S_i as small as possible.

Regarding (ii), we observe that it is a straightforward matter to generalize Propositions 1 and 2 to higher dimensions.

Proposition 1′. *Let D be a unit k-dimensional hypercube, and let n and p be positive integers such that $n^k < p$. Let $\{R_i \mid 1 \le i \le p\}$ be a collection of closed k-dimensional 'hypercuboids' contained in D whose sides are parallel to the sides of D and whose interiors are pairwise disjoint. Then there is some R_i whose shortest edge has length at most* $1/(n+1)$.

Proposition 2′. *Let D be a unit k-dimensional hypercube, and let n and p be positive integers such that $p < (n+1)^k$. Let $\{R_i \mid 1 \le i \le p\}$ be a collection of k-dimensional 'hypercuboids' whose sides are parallel to the edges of D and whose union is D. Then there is some R_i whose longest edge has length at least* $1/n$.

Propositions 1′ and 2′ are proved by induction on the number of dimensions k. The induction step is established by arguments analogous to the proofs of Propositions 1 and 2, the main difference being that we apply Theorems 1 and 2 with $m = n^{k-1}$ and $m = (n+1)^{k-1}$ respectively. Unfortunately Propositions 1′ and 2′ do not yield a solution to (ii), since the surface area of a cuboid of given volume is not determined by the length of its longest or shortest edge.

Note that Proposition 2 and its proof remain valid even if we allow each R_i to be any set whose projection on the x-axis is connected (i.e., R_i need not be rectangular). By symmetry, Proposition 2 remains valid if each R_i is a set whose projection on the y-axis is connected. In Proposition 2′ we can allow each R_i to be a set whose projections on $k-1$ of the coordinate axes are connected (the same $k-1$ axes for every R_i). Of course, 'longest edge' must be generalized to 'longest projection on an axis'.

Similarly, we can generalize Propositions 1 and 1′. In Proposition 1, each R_i can be any Cartesian product $X_i \times I_i$ where I_i is any vertical interval and X_i is any set. Symmetrically, each R_i can be any set of form $I_i \times Y_i$. In Proposition 1′ each R_i can be any product of intervals on $k-1$ of the coordinate axes with an arbitrary subset of the remaining axis (the same $k-1$ axes for every R_i). 'Shortest edge' must be generalized to 'shortest projection on an axis'.

Acknowledgments

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We are grateful to the referee for suggesting the following alternative proofs of Propositions 1 and 2, which do not depend on Dilworth's and Menger's Theorem. In Proposition 2 assume that the p rectangles are 'half' closed, so that the unit square is partitioned by them into disjoint cells. Consider the $n+1$ vertical lines $x=k/n$, $k=0,\dots,n$. If one of the rectangles meets two of these lines then its width is at least $1/n$. If no rectangles meets two lines, then the p rectangles partition the vertical lines into at most p segments, and since $p<(n+1)^2$ it follows that one of the vertical lines is partitioned into at most n segments. One of these segments has length at least $1/n$ and the corresponding rectangle has height at least $1/n$. Proposition 1 is proved similarly considering the vertical lines $x=k/(n+1)$, $k=1,\dots,n$.

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