CMSC 250 Fall 2004 — Homework 9 Answer

Due Wed., Nov. 3 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

Denote A' the complement of A and $P(A)$ the power set of A.

1. Find $P(\emptyset)$ and $P(P(P(\emptyset)))$.

   ANSWER: $P(\emptyset) = \{\emptyset\}$, $P(P(P(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

2. Prove or disprove the followings:

   (a) For all sets A and B, $(A - B) \cup (A \cap B) \subseteq A$.
   ANSWER:
   Show: $(A - B) \cup (A \cap B) \subseteq A$
   Proof:
   Let A and B be arbitrary sets.
   Let x be arbitrary in U.
   |Assume $x \in (A - B) \cup (A \cap B)$
   |$[x \in (A - B)] \vee [x \in (A \cap B)]$ by deff of union

   |Case 1: $x \in (A - B)$
   | |$x \in A \wedge x \notin B$ by def of set diff.
   | |$x \in A$ by conjunctive simplification

   |Case 2: $x \in (A \cap B)$
   | |$x \in A \wedge x \in B$
   | |$x \in A$ by conjunctive simplification

   | Therefore by dilemma $x \in A$
   $x \in (A - B) \cup (A \cap B) \rightarrow x \in A$ by CCW
   $\forall x \in U, x \in (A - B) \cup (A \cap B) \rightarrow x \in A$ by gen from GP
   Therefore, $(A - B) \cup (A \cap B) \subseteq A$ by def of subset

   (b) For all sets A and B, if $A \subseteq B$ then $B' \subseteq A'$.
   ANSWER:
   Proof:
   Let A and B be arbitrary sets.
   | Assume $A \subseteq B$
   | Let x be arbitrary in the U
   | |Assume $x \in B'$
   | | | Assume $x \notin A'$

| | | $x \in A$ by def of set comp.
| | | $x \in B$ by def of subset
| | | $x \in B \wedge x \in B'$ conj add
| | | This is a contradition
| |Assume $x \in A'$ by CCW
| $x \in B' \rightarrow x \in A'$ by CCW
| $\forall x \in U, x \in B' \rightarrow x \in A'$ by gen from GP
| $B' \subseteq A'$ by def of subset

$A \subseteq B \rightarrow B' \subseteq A'$ by CCW
$\forall A, B \in \{sets\}, A \subseteq B \rightarrow B' \subseteq A'$ by gen from GP

OR

Let A and B be arbitrary Sets.
|Assume $A \subseteq B$, which means
|$\forall x \in U, x \in A \rightarrow x \in B$ by def of subset.
|The contrapositive is:
|$\forall x \in U, x \notin B \rightarrow x \notin A$.
|That means $\forall x \in U, x \in B' \rightarrow x \in A'$ by def of set comp.
|which means $B' \subseteq A'$ by def of subset
$A \subseteq B \rightarrow B' \subseteq A'$ by CCW
Therefore, $\forall A, B \in \{sets\}, A \subseteq B \rightarrow B' \subseteq A'$ by generalizing from the GP.

(c) For all sets A,B and C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
Show: $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$
Proof:
Let A, B and C be arbitrary Sets.
Let (m, n) be arbitrary in U.
| Assume $(m, n) \in A \times (B \cap C)$.
| $m \in A \wedge n \in (B \cap C)$ by def of Cartesian Product.
| $m \in A \wedge n \in B \wedge n \in C$ by def of intersection.
| $(m \in A \wedge n \in B) \wedge (m \in A \wedge n \in C)$ by comm, assoc and idempotent
| $((m, n) \in A \times B) \wedge ((m, n) \in A \times C)$ by def of Cartesian Product.
| $(m, n) \in (A \times B) \cap (A \times C)$ by def of intersection
$(m, n) \in A \times (B \cap C) \rightarrow (m, n) \in (A \times B) \cap (A \times C)$ by CCW.
$\forall (m, n) \in U, (m, n) \in A \times (B \cap C) \rightarrow (m, n) \in (A \times B) \cap (A \times C)$ by gen from GP.
$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ by def of subset.

AND
show: $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$
proof:
Let A, B and C be arbitrary Sets.
Let (m, n) be arbitrary in U.
| Assume $(m, n) \in (A \times B) \cap (A \times C)$
| $(m, n) \in (A \times B) \wedge (m, n) \in (A \times C)$ by def of intersect
| $(m \in A \wedge n \in B) \wedge (m \in A \wedge n \in C)$ by def of cartesian product
| $m \in A \wedge (n \in B \wedge n \in C)$ by comm, assoc and idempotent

| $m \in A \land (n \in B \cap C)$ by def of intersection
| $(m, n) \in A \times (B \cap C)$ by def of cartesian product.
$(m, n) \in (A \times B) \cap (A \times C) \to (m, n) \in A \times (B \cap C)$ by CCW
$\forall (m, n) \in U, (m, n) \in (A \times B) \cap (A \times C) \to (m, n) \in A \times (B \cap C)$ by gen from GP
$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ by def of subset

Since from part 1 we know $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$
and from part 2 we know $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$
We know that $A \times (B \cap C) = (A \times B) \cap (A \times C)$ by definition of set equality.

(d) For all sets A,B and C, if $B \cap C \subseteq A$ then $(C - A) \cap (B - A) = \emptyset$.
Let A, B and C be arbitrary sets.
| Assume $B \cap C \subseteq A$
| |Assume $(C - A) \cap (B - A) \neq \emptyset$
| |$\exists m \in Z, m \in (C - A) \cap (B - A)$ by def of non-empty set
| |$m \in C - A \land m \in B - A)$ by def of intersect
| |$m \in C \land m \sim\in A \land m \in B \land m \notin A$ by def of set diff.
| |$m \in C \land m \in B$ by conj simplification.
| |$m \in C \cap B$ by def of intersection
| |$m \in A$ by def of subset
| |$m \notin A$ by conjunctive simplification
| | This is a contradiction since m can't be both in A and not in A
| $(C - A) \cap (B - A) = set$ by CCW
$B \cap C \subseteq A \to (C - A) \cap (B - A) = \emptyset$ by CCW
$\forall A, B, C \in \{sets\}, B \cap C \subseteq A \to (C - A) \cap (B - A) = \emptyset$ by gen from GP
QED

(e) For all sets A,B and C, $A - (B - C) = (A - B) \cup C$.
Counterexample: $A = \{1\}$, $B = \emptyset$ and $C = \{2\}$.

(f) For all sets A and B, $P(A \cap B) = P(A) \cap P(B)$.
Show: $P(A \cap B) \subseteq P(A) \cap P(B)$.
proof:
let x be arbitrary in U.
|Assume $x \in P(A \cap B)$
|$x \subseteq (A \cap B)$ by def of powerset.
|$x \subseteq A$ and $x \subseteq B$ by def of intersection
|$x \in P(A)$ and $x \in P(B)$ by def of powerset
|$x \in P(A) \cap P(B)$ by def of intersection
$x \in P(A \cap B) \to x \in P(A) \cap P(B)$ by CCW
$\forall x \in U, x \in P(A \cap B) \to x \in P(A) \cap P(B)$ by gen from GP.
$P(A \cap B) \subseteq P(A) \cap P(B)$ by def of subset.

Show: $P(A) \cap P(B) \subseteq P(A \cap B)$.
Proof:
Let x be arbitrary in U.
|Assume $x \in P(A) \cap P(B)$
|$x \in P(A) \land x \in P(B)$ by def of intersection

$|x \subseteq A \wedge x \subseteq B$ by def of powerset
| Let g be arbitrary in U
| $\quad$ |Assume $g \in x$.
| $\quad$ |$g \in A$ by def of subset
| $\quad$ |$g \in B$ by def of subset
| $\quad$ |$g \in A \wedge g \in B$ by conj add
| $\quad$ |$g \in A \cap B$ by def of intersect
| $g \in X \rightarrow g \in A \cap B$ by CCW
| $\forall x \in U, g \in X \rightarrow g \in A \cap B$ by gen from GP
| $x \subseteq A \cap B$ by def of subset
| $x \in P(A \cap B)$ by def of powerset
$x \in P(A) \cap P(B) \rightarrow x \in P(A \cap B)$ by CCW
$\forall x \in U, x \in P(A) \cap P(B) \rightarrow x \in P(A \cap B)$ by gen from GP
$P(A) \cap P(B) \subseteq P(A \cap B)$ by def of subset

Since $P(A \cap B) \subseteq P(A) \cap P(B)$ and $P(A) \cap P(B) \subseteq P(A \cap B)$ we have $P(A \cap B) = P(A) \cap P(B)$ by def of set equality.

3. Prove, by induction:

(a) If $A \subseteq B_i$ for all i, then $A \subseteq (B_1 \cap B_2 \cap ...)$.
Answer:
Lemma 1: $A \subseteq B$ and $A \subseteq C$ implies $A \subseteq B \cap C$.
Proof: $A \subseteq B$ and $A \subseteq C$
$\Rightarrow (x \in A \Rightarrow x \in B)$ and $(x \in A \Rightarrow x \in C)$
$\Rightarrow (x \in A \Rightarrow x \in B$ and $x \in C)$
$\Rightarrow (x \in A \Rightarrow x \in B \cap C)$
$\Rightarrow A \subseteq B \cap C$.

Show: If $A \subseteq B_i$ for all $1 \leq i \leq n$, then $A \subseteq (B_1 \cap B_2 \cap ... \cap B_n)$.
Proof: by induction on $n \geq 2$.
Base ($n = 2$): If $A \subseteq B_1$ and $A \subseteq B_2$ then $A \subseteq (B_1 \cap B_2)$ by lemme 1 above
Induction hypothesis ($n = k$): If $A \subseteq B_i$ for all $1 \leq i \leq k$, then $A \subseteq (B_1 \cap B_2 \cap ... \cap B_k)$
Induction step ($n = k + 1$):
Show: If $A \subseteq B_i$ for all $1 \leq i \leq k + 1$, then $A \subseteq (B_1 \cap B_2 \cap ... \cap B_{k+1})$
Proof:
$A \subseteq B_i$ for all $1 \leq i \leq k + 1$
$\Rightarrow A \subseteq B_i$ for all $1 \leq i \leq k$ and $A \subseteq B_{k+1}$
$\Rightarrow A \subseteq (B_1 \cap B_2 \cap ... \cap B_k)$ and $A \subseteq B_{k+1}$ (By IH)
$\Rightarrow A \subseteq (B_1 \cap B_2 \cap ... \cap B_k \cap B_{k+1})$ (by lemma 1 above)

(b) If $A_i \subseteq B$ for all i, then $(A_1 \cup A_2 \cup ...) \subseteq B$.
Answer:
Lemma 2: $A \subseteq C$ and $B \subseteq C$ implies $A \cup B \subseteq C$.
Proof: $A \subseteq C$ and $B \subseteq C$
$\Rightarrow (x \in A \Rightarrow x \in C)$ and $(x \in B \Rightarrow x \in C)$ by def of subset
$\Rightarrow (x \notin C \Rightarrow x \notin A$ and $x \notin B)$ by by contrapositive and distributive
$\Rightarrow (x \notin C \Rightarrow x \in A' \cap B')$ by def of intersection

$\Rightarrow (x \notin A' \cap B' \Rightarrow x \in C)$ by contrapositive
$\Rightarrow (x \in A \cup B \Rightarrow x \in C)$ by DeMorgan's
$\Rightarrow A \cup B \subseteq C$ by def of subset

Show: If $A_i \subseteq B$ for all $1 \leq i \leq n$, then $(A_1 \cup A_2 \cup ... \cup A_n) \subseteq B$.
Proof: by induction on $n \geq 2$.
Base ($n = 2$): If $A_1 \subseteq B$ and $A_2 \subseteq B$ then $(A_1 \cup A_2) \subseteq B$ by Lemma 2 above
Induction hypothesis ($n = k$): If $A_i \subseteq B$ for all $1 \leq i \leq k$, then $(A_1 \cup A_2 \cup ... \cup A_k) \subseteq B$.
Induction step ($n = k + 1$):
Show: If $A_i \subseteq B$ for all $1 \leq i \leq k + 1$, then $(A_1 \cup A_2 \cup ... \cup A_{k+1}) \subseteq B$.
Proof:
$A_i \subseteq B$ for all $1 \leq i \leq k + 1$
$\Rightarrow A_i \subseteq B$ for all $1 \leq i \leq k$ and $A_{k+1} \subseteq B$
$\Rightarrow (A_1 \cup A_2 \cup ... \cup A_k) \subseteq B$ and $A_{k+1} \subseteq B$ (by IH)
$\Rightarrow (A_1 \cup A_2 \cup ... \cup A_{k+1}) \subseteq B$ (by Lemma 2 above