Name (PRINTED): ______________________

Student ID #: ______________________

Section # (or TA's: name and time) ______________________

CMSC 250 Quiz #9 ANSWERS Wed., Oct. 27, 2004

Write all answers legibly in the space provided. The number of points possible for each question is indicated in square brackets – the total number of points on the quiz is 30, and you will have exactly **20** minutes to complete this quiz. You may not use calculators, textbooks or any other aids during this quiz.

1. [15 pnts.]Use regular induction to prove the following inequality.

$$\forall n \in Z^{\geq 2},\ \sum_{r=2}^{n}(r^2 - r) < \frac{n^3}{3}$$

Base Case: (n=2)
LHS: $\sum_{r=2}^{2}(r^2 - r) = 2^2 - 2 = 4 - 2 = 2$
RHS: $\frac{2^3}{3} = \frac{8}{3} = 2\frac{2}{3}$
$LHS \leq RHS$ so the statement is true when $n = 2$

Inductive Hypothesis:(n=x)
$\sum_{r=2}^{x}(r^2 - r) < \frac{x^3}{3}$

Inductive Step:(n=x+1)
show: $\sum_{r=2}^{x+1}(r^2 - r) < \frac{(x+1)^3}{3}$

proof:
PART 1 (find b):
$\sum_{r=2}^{x}(r^2 - r) < \frac{x^3}{3}$ by the IH
$\sum_{r=2}^{x}(r^2 - r) + \sum_{r=x+1}^{x+1}(r^2 - r) < \frac{x^3}{3} + \sum_{r=x+1}^{x+1}(r^2 - r)$ by adding the same summation to both sides
$\sum_{r=2}^{x+1}(r^2 - r) < \frac{x^3}{3} + ((x+1)^2 - (x+1))$ by combining and expanding the summation
$\sum_{r=2}^{x+1}(r^2 - r) < \frac{x^3}{3} + (x^2 + 2x + 1 - x - 1)$ by squaring the polynomial
$\sum_{r=2}^{x+1}(r^2 - r) < \frac{x^3}{3} + (x^2 + x)$ by combining like terms
Let $b = \frac{x^3}{3} + x^2 + x$

PART 2 (show that $b \leq \frac{(x+1)^3}{3}$ in other words that $\frac{x^3}{3} + x^2 + x \leq \frac{(x+1)^3}{3}$)

Assume $\frac{x^3}{3} + x^2 + x > \frac{(x+1)^3}{3}$
$\frac{x^3+3x^2+3x}{3} > \frac{x^3+3x^2+3x+1}{3}$ by putting things over a common denominator and cubing the polynomial
$\frac{x^3+3x^2+3x}{3} > \frac{x^3+3x^2+3x}{3} + \frac{1}{3}$ by splitting up the fraction
$0 > \frac{1}{3}$ by subtracting the same quantity from both sides
This is a contradiction,
so our assumption must be false and we know for sure then that
$\frac{x^3}{3} + x^2 + x \leq \frac{(x+1)^3}{3}$ by closing the conditional world

We know from part 1 that $\sum_{r=2}^{x+1}(r^2 - r) < \frac{x^3}{3} + x^2 + x$
Therefore by the transitive property of inequalities, we know
$\sum_{r=2}^{x+1}(r^2 - r) \leq \frac{(x+1)^3}{3}$
QED

OR AN ALTERNATIVE PROOF PORTION:

proof:
$\sum_{r=2}^{x}(r^2 - r) < \frac{x^3}{3}$ by the IH
$\sum_{r=x+1}^{x+1}(r^2 - r) \leq x^2 + x + \frac{1}{3}$ by LEMMA #1 proved below
$\sum_{r=2}^{x}(r^2 - r) + \sum_{r=x+1}^{x+1}(r^2 - r) < \frac{x^3}{3} + x^2 + x + \frac{1}{3}$ by adding inequalities
$\sum_{r=2}^{x+1}(r^2 - r) < \frac{x^3+3x^2+3x+1}{3}$ by combining summations and putting terms over a common demoninator
$\sum_{r=2}^{x+1}(r^2 - r) < \frac{(x+1)^3}{3}$ by factorring the polynomial
QED

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LEMMA #1
This lemma needs to prove that $\sum_{r=x+1}^{x+1}(r^2 - r) \leq x^2 + x + \frac{1}{3}$

Assume $\sum_{r=x+1}^{x+1}(r^2 - r) > x^2 + x + \frac{1}{3}$
$((x+1)^2 - (x+1) > x^2 + x + \frac{1}{3}$ by expanding the summation
$x^2 + 2x + 1 - x - 1 > x^2 + x + \frac{1}{3}$ by squaring the polynomial
$x^2 + x > x^2 + x + \frac{1}{3}$ by combining like terms
$0 > \frac{1}{3}$ by subtracting $x^2 + x$ from both sides
This is a contradiction because $\frac{1}{3}$ is clearly non-negative.
Since we have reached a contradiction, we know our assumption must be false.

Therefore $\sum_{r=x+1}^{x+1}(r^2 - r) \leq x^2 + x + \frac{1}{3}$ by closing the conditional world

2. [15 pnts.] Use strong induction to prove the following statement.

Assume:

$$a_0 = 4, \quad a_1 = 9, \quad \text{and} \quad \forall n \in Z^{\geq 2} a_n = 6a_{n-1} - 9a_{n-2}$$

Prove that

$$\forall k \in Z^{\geq 0}, a_k = (4-k)3^k$$

Base Case: (n=0, n=1)
n=0: $a_0 = 4$ and $(4-0)3^0 = 4(1) = 4$
n=1: $a_1 = 9$ and $(4-1)3^1 = (3)(3^1) = 9$
The formula holds for both base cases.

Inductive Hypothesis:(n=i $\forall i \in Z, 0 \leq i < p$)
$a_i = (4-i)3^i$

Inductive Step:(n=p)
show: $a_p = (4-p)3^p$

proof:
$a_p = 6a_{p-1} - 9a_{p-2}$
Since $p-1 < p$ and $p-2 < p$ because we are subtracting positive values on the smaller side
And since $p-1 \geq 0$ and $p-2 \geq 0$ $\forall p \in Z^{\geq 2}$,
We know we can apply the IH to both a_{p-1} and to a_{p-2}.
Doing this we get the facts that:
$a_{p-1} = (4-(p-1))3^{p-1} = (5-p)3^{p-1}$
and $a_{p-2} = (4-(p-2))3^{p-2} = (6-p)3^{p-2}$
By substitution into the above equation using these we get $a_p = 6[(5-p)3^{p-1}] - 9[(6-p)3^{p-2}]$
$a_p = 2[(5-p)3^p] - [(6-p)3^p]$ by combining factors of 3
$a_p = 3^p[2(5-p) - (6-p)]$ by distributing out the 3 term
$a_p = 3^p[10 - 2p - 6 + p]$ by multiplying through the 2
$a_p = 3^p[4-p]$ by combining like terms
$a_p = (4-p)3^p$ by commutativity of multiplication
QED