Counting Elements in a List

- How many integers in the list from 1 to 10?

- How many integers in the list from m to n?
(assuming m <= n)

How many in a list divisible by something:

- How many positive three digit integers are there?
  - (this means only the ones that require 3 digits)
  - 999 – 100 + 1 = 900
- How many three digit integers are divisible by 5?
  - think about the definition of divisible by

    $x \mid y \leftrightarrow \exists\, k \in Z,\ y = kx$ and then count the k's that work

  100, 101, 102, 103, 104, 105, 106,… 994, 995, 996, 997, 998, 999

  20*5 21*5 … 199*5
  - count the integers between 20 and 199
  - 199 – 20 + 1 = 180

Probability

likelihood of a specific event

- Sample Space = set of all possible outcomes
- Event = subset of sample space
- Equal Probability Formula:
  - Given a finite sample space S where all outcomes are equally likely
  - Select an event E from the sample space S
  - The probability of event E from sample space S:

$$P(E) = \frac{n(E)}{n(S)}$$

Flipping Two Coins

- Sample Space = {(H,H), (H,T), (T,H), (T,T)}
- Probability of no heads
- Probability of at least one head
- Probability of same sides on the two coins
- Note: probability & actual outcomes often differ

Standard Playing Cards

- values: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- suits: D(♦), H(♥), S(♠), C(♣)

- probability of drawing the Ace of Spades
- probability of drawing a Spade
- probability of drawing a face card
- probability of drawing a red face card

Rolling Two Six-Sided Dice

- Sample Space

{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
…
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

- Probability of rolling a 10
- Probability of rolling a pair

Multi-level Probability

- If I toss a coin once – the probability of Head = ½
- If I toss that coin 5 times
  - the probability of all heads

$$\frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{2^5}$$

  - the probability of exactly 4 heads

$$\frac{5}{2^5}$$

Multiplication Rule

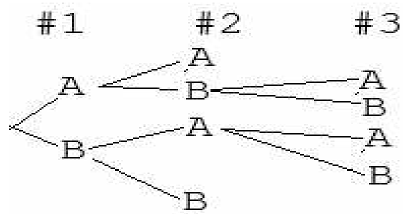
- 1^{st} step can be performed n_1 ways
- 2^{nd} step can be performed n_2 ways
- …
- K^{th} step can be performed n_k ways

- operation can be performed $n_1 * n_2 * \ldots * n_k$ ways

- Cartesian product n(A)=3, n(B)=2, n(C)=4
  - n(AxBxC) = 24
  - n(AxB) = 6 n((AxB)xC) = 24

Tournament Play

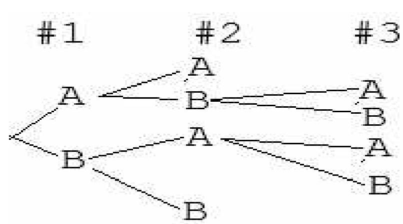
- Team A and Team B in "Best of 3" Tournament
- where they each have an equal likelihood of winning each game



  - Do leaves add up to 1?
  - Do we have to play 3 games?
  - Do A and B have an equal chance of winning?

What if A wins 2 of every 3 games?

- Each line for A must have a 2/3
- Each line for B must have a 1/3



- How likely is A to win the tournament?
- How likely is B to win the tournament?

Using the Multiplication Rule for Selecting a PIN

- Number of 4 digit PINs of (0,1,2,.)
  - with repetition allowed = 4*4*4*4=256
  - with no repetition allowed = 4*3*2*1=24
- Extra Rules :
  - . (the period) can't be first or last
  - 0 can't be first
    - with repetition allowed = 2*4*4*3
    - without repetition allowed = 2*2*2*1

Probabilities with PINs

- Number of 4 digit PINs of (0,1,2,.)
  - with repetition allowed = 4*4*4*4=256
  - with no repetition allowed = 4*3*2*1=24
- What is the probability that your 4 digit PIN has no repeated characters?
- What is the probability that your 4 digit PIN does have repeated characters?
- probability of the complement of an event

$$P(E') = P(E^c) = 1-P(E)$$

Difference Rule Formally

- If A is a finite set and B $\subseteq$A, then
  n(A-B) = n(A) – n(B)

- One Application:

probability of the complement of an event

P(E') = P(E^c) = 1-P(E)

PINs with less specified length Addition Rule

- Assume it can be a 2,3 or 4 length PIN

Partition the problem

number of 2 length PINs w/rep allowed: 4*4 = 16

number of 3 length PINs w/rep allowed: 4*4*4 = 64

number of 4 length PINs w/rep allowed: 4*4*4*4 = 256

Number PINs if allowing length of 2,3 or 4 = 336

Addition Rule Formally

- if $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = A$
- and $A_1, A_2, A_3, \ldots, A_k$ are pairwise disjoint

in other words, if these subsets form a partition of A

$n(A) = n(A_1)+n(A_2)+n(A_3)+\ldots+n(A_k)$

Another example for Multiplication Rule and Addition Rule

- How many 3 digit integers are divisible by 5?
  - How many end in a 0? 9*10*1 = 90
  - How many end in a 5? 9*10*1 = 90
  - These form a partition with the set of numbers divisible by 5 so
  - 90 + 90 = 180

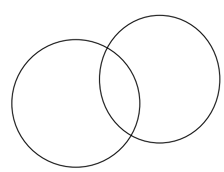
Where Multiplication Rule Doesn't Work

- People= {Angel, Bob, Carol, Dan}
- need to be appointed as
  - president, vice-president, and treasurer
  - nobody can hold more than one office
  - Angel doesn't want to be president
  - Only Bob and Dan want to be vice-president

Inclusion/Exclusion Rule

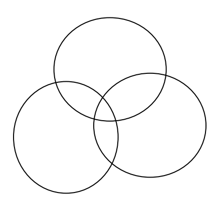
- If there are two sets:

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$



- If there are three sets:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$



Permutations

- Different ways of arranging objects
  - in a line or circle
  - without duplication/ all items distinguishable
  - note: order is taken into account
- Number of linear permutations of N objects = N!

  N possible for 1st position * (N-1) for 2nd * …* (1) for last
- Number of circular permutations of N objects = (N-1)!

  Fix one person,

  then (N-1) possible for next position * (N-2) for 2nd * …* (1) for last

r-Permutations

If there are n things in the set,
and you want to line-up only r of them.

$$P(n,r) = {}_rP_n = \frac{n!}{(n-r)!}$$

- Example:

Class = {Alice, Bob, Carol, Dan}

  - select a president and a vice president to represent the class

Combinations

- Different ways of selecting objects
  - Counting Subsets
  - without duplication/ all items distinguishable
  - note: <u>order is not taken into account</u>

$$\forall n, r \in Z^{\geq 0} where\ n \geq r,$$

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$$

- Examples:

Class = {Alice, Bob, Carol, Dan}
  - select two class representatives
  - select three class representatives

Harder Examples selecting "class representatives"

Class = {Alice,Bob,Carol,Dan, Erin, Fred}

- select 2 – no restrictions
- select 2 – assuming Alice and Bob must stay together
- select 3 – no restrictions
- select 3 – assuming Alice and Bob must stay together
- select 3 – assuming Alice and Bob refuse to serve together

Different Types of Members

{Alice, Bob, Carol, Dan, Erin, Fred, George, Harry}
pink names are girls and blue names are boys

- 8 people in the set: 3 girls & 5 boys

make a 5 member team of 2 girls and 3 boys
make a 5 member team that has only one girl
make a 5 member team that has no girls
make a 5 member team that has at least one girl

Permutations but of indistinguishable items

- Assume you have a set of 15 beads
  - 3 Red
  - 2 Black
  - 4 orange
  - 6 green
- Select positions of R's, then B's, then O's then G's

$$\binom{15}{3} * \binom{12}{2} * \binom{10}{4} * \binom{6}{6} = \frac{15!}{3!2!4!6!}$$

Combinations with Repetition

- {a,b,c,d,e}
- How many 3-combinations can I make without repetition?
- How many 3-combinations can I make with unlimited repetition allowed?

- these are multisets [a,b,c]
  - not sets {a,b,c}
  - not tuples (a,b,c)

Probability with Combinations

- Assume there are 32 people in the class
- And that 7 will be chosen to get extra homework
- What is the probability that you get extra homework

- Number of ways to select the "lucky 7"
- Number of ways to select if "I get HW"

- P(I get HW)

Properties of r-Permutations and proofs of those properties

P(n,1) = n

P(n,2) = n^2 -n

P(n,2) + P(n,1) = n^2

P(n,n) = n!

P(n,n-1) = n!

Properties of Combinations and their proofs

$\binom{n}{0}=1$	$\binom{n}{n}=1$
$\binom{n}{1}=n$	$\binom{n}{n-1}=n$
$\binom{n}{2}=\frac{n(n-1)}{2}$	$\binom{n}{n-2}=\frac{n(n-1)}{2}$

$$\binom{n}{r}=\binom{n}{n-r}$$

Binomial Theorem

- $(x+y)^2$
- $(x+y)^3$

…

- $(x+y)^n$

$$\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n-1},\binom{n}{n}$$

Notice Similarities

- The number of Non-negative Integer Solutions of the equation

$$x_1 + x_2 + \ldots + x_n = r \qquad x_i \geq 0, \forall i \in Z^{1 \leq i \leq n}$$

- The number of selections, with repetition, of size r from a collection of size n.

- The number of ways r identical objects can be distributed among n distinct containers.

Conditional Probability

- Probability of B given that A is known to have happened

$$P(B \mid A) = \frac{P(A \bigcap B)}{P(A)}$$

- If P(B) = P(B|A) then

  event B is Independent of event A