

Terminology

- Domain: set which holds the values to which we apply the function
- Co-domain: set which holds the possible values (results) of the function
- Range: set of actual values received when applying the function to the values of the domain

Function

- A “total” function is a relationship between elements of the domain and elements of the co-domain where each and every element of the domain relates to one and only one value in the co-domain
- A “partial” function does not need to map every element of the domain.
- $f: X \rightarrow Y$
 - f is the function name
 - X is the domain
 - Y is the co-domain
 - $x \in X \quad y \in Y \quad f$ sends x to y
 - $f(x) = y$ f of x ; value of f at x ; image of x under f

Formal Definitions

- Range of f is $\{y \in Y \mid \exists x \in X, f(x) = y\}$
 - where X is the domain and Y is the co-domain
- Inverse image of y is $\{x \in X \mid f(x) = y\}$
 - the set of things that map to y
- Arrow Diagrams
 - Determining if they are functions using the Arrow Diagram

Terminology of Functions

- Equality of Functions

$\forall f, g \in \{\text{functions with the same Domain and Co Domain}\},$
 $f = g \leftrightarrow \forall x \in X, f(x) = g(x)$

- F is a One-to-One (or Injective) Function iff

$$\forall x_1, x_2 \in X \quad F(x_1) = F(x_2) \rightarrow x_1 = x_2$$

$$\forall x_1, x_2 \in X \quad x_1 \neq x_2 \rightarrow F(x_1) \neq F(x_2)$$

- F is NOT a One-to-One Function iff

$$\exists x_1, x_2 \in X, (F(x_1) = F(x_2)) \wedge (x_1 \neq x_2)$$

- F is an Onto (or Surjective) Function iff

$$\forall y \in Y \quad \exists x \in X, F(x) = y$$

- F is NOT an Onto Function iff

$$\exists y \in Y \quad \forall x \in X, F(x) \neq y$$

Proving functions one-to-one and onto

$$f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = 3x - 4$$

- Prove or give a counter example that f is one-to-one

use
def: $\forall x_1, x_2 \in \mathbb{R}, f(x_1) = f(x_2) \rightarrow x_1 = x_2$

- Prove or give a counter example that f is onto

use
def: $\forall y \in \mathbb{R} \exists x \in \mathbb{R}, f(x) = y$

One-to-One Correspondence or Bijection

$F: X \rightarrow Y$ is bijective $\leftrightarrow F: X \rightarrow Y$ is one-to-one & onto

- $F: X \rightarrow Y$ is bijective $\leftrightarrow$ It has an inverse function

$$\exists F^{-1}: Y \rightarrow X$$

$$\forall x \in X, F(x) = y \rightarrow F^{-1}(y) = x$$

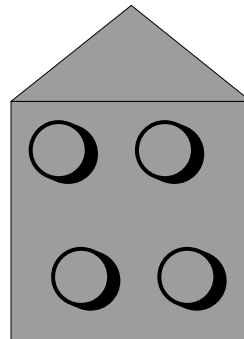
$$\forall y \in Y, F^{-1}(y) = x \rightarrow F(x) = y$$

Proving something is a bijection

- $F: \mathbb{Z} \rightarrow \mathbb{Z}$ $F(x) = x + 1$
- Prove it is one-to-one
- Prove it is onto
- Then it is a bijection

- So it has an inverse function
 - find F^{-1}

Pigeonhole Principle



- **Basic Form**

A function from one finite set to a smaller finite set cannot be one-to-one; there must be at least two elements in the domain that have the same image in the co-domain.

Examples

- Using this class as the domain,
Must two people share a birthmonth?
Must two people share a birthday?
- $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
If I select 5 integers at random from this set,
must two of the numbers sum to 9?
If I select 4 integers?

Other (more useful) Forms of the Pigeonhole Principle

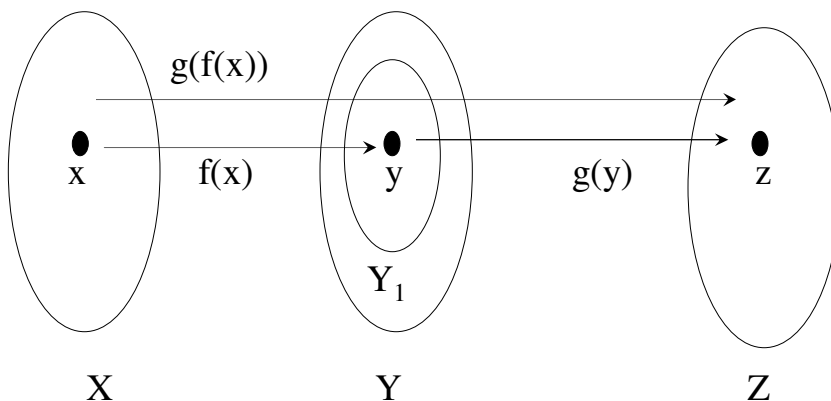
- Generalized Pigeonhole Principle
 - For any function f from a finite set X to a finite set Y and for any positive integer k , if $n(X) > k \cdot n(Y)$, then there is some $y \in Y$ such that y is the image of at least $k+1$ distinct elements of X .
- Contrapositive Form of Generalized Pigeonhole Principle
 - For any function f from a finite set X to a finite set Y and for any positive integer k , if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most $k \cdot n(Y)$ elements.

Examples

- Using Generalized Form:
 - Assume 50 people in the room, how many must share the same birthmonth?
 - $n(A)=5$ $n(B)=3$ $F:P(A) \rightarrow P(B)$
How many elements of $P(A)$ must map to a single element of $P(B)$?
- Using Contrapositive of the Generalized Form:
 - $G:X \rightarrow Y$ where Y is the set of 2 digit integers that do not have distinct digits. Assuming no more than 5 elements of X can map to a single element of Y , how big can X be?
 - You have 5 busses and 100 students. No bus can carry over 25 students. Show that at least 3 busses must have over 15 students each.

Composition of Functions

- $f:X \rightarrow Y_1$ and $g:Y \rightarrow Z$ where $Y_1 \subseteq Y$
 - $g \circ f: X \rightarrow Z$ where $\forall x \in X, g(f(x)) = g \circ f(x)$



Composition on finite sets

- Example

– $X = \{1,2,3\}$ $Y1 = \{a,b,c,d\}$ $Y = \{a,b,c,d,e\}$ $Z = \{x,y,z\}$

$f(1)=c$	$g(a)=y$	$g \circ f(1)=g(f(1))=z$
$f(2)=b$	$g(b)=y$	$g \circ f(2)=g(f(2))=y$
$f(3)=a$	$g(c)=z$	$g \circ f(3)=g(f(3))=y$
	$g(d)=x$	
	$g(e)=x$	

Composition for infinite sets

$$f: \mathbb{Z} \rightarrow \mathbb{Z} \quad f(n)=n+1$$

$$g: \mathbb{Z} \rightarrow \mathbb{Z} \quad g(n)=n^2$$

$$g \circ f(n)=g(f(n))=g(n+1)=(n+1)^2$$

$$f \circ g(n)=f(g(n))=f(n^2)=n^2+1$$

note: $g \circ f \neq f \circ g$

Identity function

- i_X the identity function for the domain X
 $i_X : X \rightarrow X \quad \forall x \in X, i_X(x) = x$
- i_Y the identity function for the domain Y
 $i_Y : Y \rightarrow Y \quad \forall y \in Y, i_Y(y) = y$
- composition with the identity functions

Composition of a function with its inverse function

- $f \circ f^{-1} = i_Y$
- $f^{-1} \circ f = i_X$
- Composing a function with its inverse returns you to the starting place.
(note: $f: X \rightarrow Y$ and $f^{-1}: Y \rightarrow X$)

One-to-One in Composition

- If $f:X \rightarrow Y$ and $g:Y_1 \rightarrow Z$ are both one-to-one, then $g \circ f:X \rightarrow Z$ is one-to-one when $Y = Y_1$
- If $f:X \rightarrow Y$ and $g:Y_1 \rightarrow Z$ are both onto, then $g \circ f:X \rightarrow Z$ is onto when $Y = Y_1$

Cardinality

- Comparing the “sizes” of sets
 - finite sets ($\emptyset$ or has a bijective function from it to $\{1,2,\dots,n\}$)
 - infinite sets (can't have a bijective function from it to $\{1,2,\dots,n\}$)
- $\forall A,B \in \{\text{sets}\}$, A and B have the same cardinality $\leftrightarrow$ there is a one-to-one correspondence from A to B

In other words,

$\text{Card}(A) = \text{Card}(B) \leftrightarrow \exists f \in \{\text{functions}\}, f:A \rightarrow B \wedge f \text{ is a bijection}$

Countability of sets of integers

- $\mathbb{Z}^+$ is a countably infinite set
- $\mathbb{Z}^{\geq 0}$ is a countably infinite set
- $\mathbb{Z}$ is a countably infinite set
- $\mathbb{Z}^{\text{even}}$ is also a countably infinite set

$$\text{Card.}(\mathbb{Z}^+) = \text{Card.}(\mathbb{Z}^{\geq 0}) = \text{Card.}(\mathbb{Z}) = \text{Card.}(\mathbb{Z}^{\text{even}})$$

Real Numbers

- We'll take just a part of this infinite set
- Reals between 0 and 1 (non-inclusive)
 - $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$
- All elements of X can be written as
 - $0.a_1a_2a_3\dots a_n\dots$

Cantor's Proof

- Assume the set $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is countable
- Then the elements in the set can be listed

$0.a_{11}a_{12}a_{13}a_{14}\dots a_{1n}\dots$
 $0.a_{21}a_{22}a_{23}a_{24}\dots a_{2n}\dots$
 $0.a_{31}a_{32}a_{33}a_{34}\dots a_{3n}\dots$
 $\dots \dots \dots$
- Select the digits on the diagonal
- build a number

$$d_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$$
- d differs in the n^{th} position from the n^{th} in the list

All Reals

- $\text{Card.}(\{x \in \mathbb{R} \mid 0 < x < 1\}) = \text{Card.}(\mathbb{R})$

Positive Rationals $\mathbb{Q}^+$

- $\text{Card.}(\mathbb{Q}^+) = ? = \text{Card.}(\mathbb{Z})$

log function properties (from back cover of textbook)

definition of log: $\log_b x = y \leftrightarrow b^y = x$

$$\log_b b = 1$$

$$\log_b x^a = a * \log_b x$$

$$\log_b x = \frac{\log_n x}{\log_n b}$$

$$\log_b (x * y) = \log_b x + \log_b y$$

$$\log_b \left(\frac{x}{y} \right) = \log_b x - \log_b y$$

$$\log_b (x) = \log_b y \rightarrow x = y$$