Terminology

- Domain: set which holds the values to which we apply the function
- Co-domain: set which holds the possible values (results) of the function
- Range: set of actual values received when applying the function to the values of the domain

Function

- A "total" function is a relationship between elements of the domain and elements of the co-domain where each and every element of the domain relates to one and only one value in the co-domain
- A "partial" function does not need to map every element of the domain.
- f: $X \rightarrow Y$
  - f is the function name
  - X is the domain
  - Y is the co-domain
  - $x \in X \quad y \in Y$ f sends x to y
  - f(x) = y f of x ; value of f at x ; image of x under f

Formal Definitions

- Range of f is $\{y \in Y \mid \exists x \in X, f(x) = y\}$
  - where X is the domain and Y is the co-domain
- Inverse image of y is $\{x \in X \mid f(x) = y\}$
  - the set of things that map to y
- Arrow Diagrams
  - Determining if they are functions using the Arrow Diagram

Terminology of Functions

- Equality of Functions

$\forall f,g \in \{$functions with the same Domain and Co Domain$\}$, $f=g \leftrightarrow \forall x \in X, f(x)=g(x)$

- F <u>is a One-to-One</u> (or Injective) Function iff

$\forall x_1,x_2 \in X \quad F(x_1) = F(x_2) \rightarrow x_1=x_2$

$\forall x_1,x_2 \in X \quad x_1 \neq x_2 \rightarrow F(x_1) \neq F(x_2)$

- F <u>is NOT a One-to-One</u> Function iff

$\exists x_1,x_2 \in X, (F(x_1) = F(x_2)) \wedge (x_1 \neq x_2)$

- F <u>is an Onto</u> (or Surjective) Function iff

$\forall y \in Y \quad \exists x \in X, F(x) = y$

- F <u>is NOT an Onto</u> Function iff

$\exists y \in Y \ \forall x \in X, F(x) \neq y$

Proving functions one-to-one and onto

f:R→R f(x)=3x-4

- Prove or give a counter example that f is one-to-one

use def: $\forall x_1, x_2 \in R, f(x_1) = f(x_2) \rightarrow x_1 = x_2$

- Prove or give a counter example that f is onto

use def: $\forall y \in R \exists x \in R, f(x) = y$

One-to-One Correspondence or Bijection

F:X →Y is bijective ↔ F:X →Y is one-to-one & onto

- F:X →Y is bijective ↔ It has an inverse function

$$\exists F^{-1} : Y \rightarrow X$$

$$\forall x \in X, F(x) = y \rightarrow F^{-1}(y) = x$$

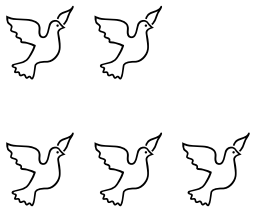
$$\forall y \in Y, F^{-1}(y) = x \rightarrow F(x) = y$$

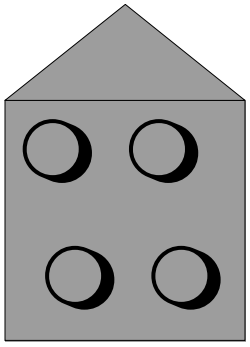
Proving something is a bijection

- $F:Z \rightarrow Z \quad F(x)=x+1$
- Prove it is one-to-one
- Prove it is onto
- Then it is a bijection

- So it has an inverse function
  - find F^{-1}

Pigeonhole Principle





- Basic Form

  A function from one finite set to a smaller finite set cannot be one-to-one; there must be at least two elements in the domain that have the same image in the co-domain.

Examples

- Using this class as the domain,

Must two people share a birthmonth?

Must two people share a birthday?

- A = {1,2,3,4,5,6,7,8}

If I select 5 integers at random from this set, must two of the numbers sum to 9?

If I select 4 integers?

Other (more useful) Forms of the Pigeonhole Principle

- Generalized Pigeonhole Principle
  - For any function f from a finite set X to a finite set Y and for any positive integer k, if $n(X) > k*n(Y)$, then there is some $y \in Y$ such that y is the image of at least k+1 distinct elements of X.
- Contrapositive Form of Generalized Pigeonhole Principle
  - For any function f from a finite set X to a finite set Y and for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most $k*n(Y)$ elements.

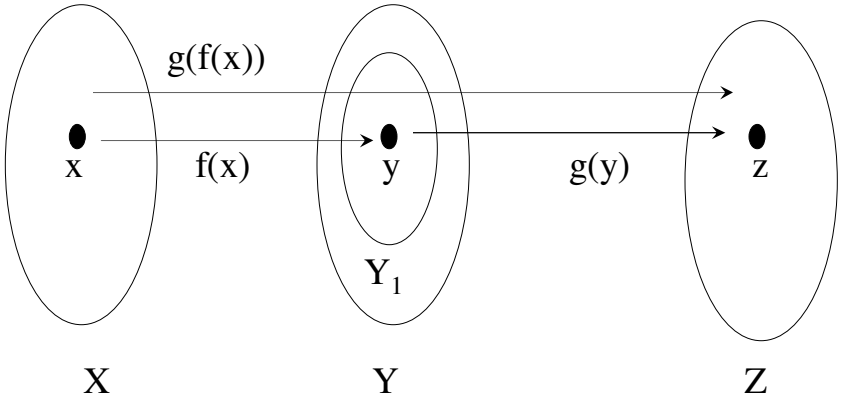
Examples

- Using Generalized Form:
  - Assume 50 people in the room, how many must share the same birthmonth?
  - n(A)=5 n(B)=3 F:P(A)→P(B)

    How many elements of P(A) must map to a single element of P(B)?
- Using Contrapositive of the Generalized Form:
  - G:X→Y where Y is the set of 2 digit integers that do not have distinct digits. Assuming no more than 5 elements of X can map to a single element of Y, how big can X be?
  - You have 5 busses and 100 students. No bus can carry over 25 students. Show that at least 3 busses must have over 15 students each.

Composition of Functions

- $f:X \rightarrow Y_1$ and $g:Y \rightarrow Z$ where $Y_1 \subseteq Y$
  - $g \circ f:X \rightarrow Z$ where $\forall x \in X, g(f(x)) = g \circ f(x)$



Composition on finite sets

- Example
  - X = {1,2,3} Y1 = {a,b,c,d} Y={a,b,c,d,e} Z = {x,y,z}

f(1)=c	g(a)=y	g∘f(1)=g(f(1))=z
f(2)=b	g(b)=y	g∘f(2)=g(f(2))=y
f(3)=a	g(c)=z	g∘f(3)=g(f(3))=y
	g(d)=x	
	g(e)=x	

Composition for infinite sets

$f:Z \rightarrow Z \quad f(n)=n+1$

$g:Z \rightarrow Z \quad g(n)=n^2$

$g \circ f(n)=g(f(n))=g(n+1)=(n+1)^2$

$f \circ g(n)=f(g(n))=f(n^2)=n^2+1$

note: $g \circ f \neq f \circ g$

Identity function

- i_X the identity function for the domain X
  $i_X : X \to X \quad \forall x \in X, i_X(x) = x$
- i_Y the identity function for the domain Y
  $i_Y : Y \to Y \quad \forall y \in Y, i_Y(y) = y$
- composition with the identity functions

Composition of a function with its inverse function

- $f \circ f^{-1} = i_Y$
- $f^{-1} \circ f = i_X$
- Composing a function with its inverse returns you to the starting place.

  (note: $f: X \to Y$ and $f^{-1}: Y \to X$)

One-to-One in Composition

- If $f:X \to Y$ and $g:Y_1 \to Z$ are both one-to-one, then $g \circ f:X \to Z$ is one-to-one when $Y = Y_1$
- If $f:X \to Y$ and $g:Y_1 \to Z$ are both onto, then $g \circ f:X \to Z$ is onto when $Y = Y_1$

Cardinality

- Comparing the "sizes" of sets
  - finite sets ($\varnothing$ or has a bijective function from it to $\{1,2,\ldots,n\}$)
  - infinite sets (can't have a bijective function from it to $\{1,2,\ldots,n\}$)
- $\forall A,B \in \{\text{sets}\}$, A and B have the same cardinality $\leftrightarrow$ there is a one-to-one correspondence from A to B

In other words,

$\text{Card}(A) = \text{Card}(B) \leftrightarrow \exists f \in \{\text{functions}\}, f:A \to B \wedge f$ is a bijection

Countability of sets of integers

- Z^{+} is a countably infinite set
- $Z^{\geq 0}$ is a countably infinite set
- Z is a countably infinite set
- Z^{even} is also a countably infinite set

$Card.(Z^{+})=Card.(Z^{\geq 0})=Card.(Z)=Card.(Z^{even})$

Real Numbers

- We'll take just a part of this infinite set
- Reals between 0 and 1 (non-inclusive)
  - $X = \{x \in R \mid 0<x<1\}$
- All elements of X can be written as
  - $0.a_1a_2a_3\ldots a_n\ldots$

Cantor's Proof

- Assume the set X = $\{x \in R | 0<x<1\}$ is countable
- Then the elements in the set can be listed

  $0.a_{11}a_{12}a_{13}a_{14}\ldots a_{1n}\ldots$

  $0.a_{21}a_{22}a_{23}a_{24}\ldots a_{2n}\ldots$

  $0.a_{31}a_{32}a_{33}a_{34}\ldots a_{3n}\ldots$

  … … … …
- Select the digits on the diagonal
- build a number

$$d_n = \begin{cases} 1 & if\ a_{nn} \neq 1 \\ 2 & if\ a_{nn} = 1 \end{cases}$$

- d differs in the n^{th} position from the n^{th} in the list

All Reals

- Card.($\{x \in R | 0<x<1\}$) = Card.(R)

Positive Rationals Q^+

- Card.(Q^+) =?= Card.(Z)

log function properties (from back cover of textbook)

definition of log: $\log_b x = y \leftrightarrow b^y = x$

$\log_b b = 1$	$\log_b x = \frac{\log_n x}{\log_n b}$
$\log_b x^a = a * \log_b x$	
$\log_b (x * y) = \log_b x + \log_b y$	
$\log_b (\frac{x}{y}) = \log_b x - \log_b y$	
$\log_b (x) = \log_b y \rightarrow x = y$	