Proof Must Have

- Statement of what is to be proven.
- "Proof:" to indicate where the proof starts
- Clear indication of flow
- Clear indication of reason for each step
- Careful notation, completeness and order
- Clear indication of the conclusion

Number Theory - Ch 3 Definitions

- **Z** --- integers
- **Q** - rational numbers (quotients of integers)
  - $r \in \mathbf{Q} \leftrightarrow \exists a,b \in \mathbf{Z}, (r = a/b) \wedge (b \neq 0)$
- Irrational = not rational
- **R** --- real numbers
- superscript of $^+$ --- positive portion only
- superscript of $^-$ --- negative portion only
- other superscripts: Z^{even}, Z^{odd}, $Q^{>5}$

- "closure" of these sets for an operation
  - Z closed under what operations?

Integer Definitions

- even integer
  - $n \in \mathbf{Z}^{even} \leftrightarrow \exists k \in \mathbf{Z}, n = 2k$
- odd integer
  - $n \in \mathbf{Z}^{odd} \leftrightarrow \exists k \in \mathbf{Z}, n = 2k+1$
- prime integer ($Z^{>1}$)
  - $n \in \mathbf{Z}^{prime} \leftrightarrow \forall r,s \in \mathbf{Z}^{+}, (n=r*s) \rightarrow (r=1) v (s=1)$
- composite integer ($Z^{>1}$)
  - $n \in \mathbf{Z}^{composite} \leftrightarrow \exists r,s \in \mathbf{Z}^{+}, n=r*s \wedge (r \neq 1) \wedge (s \neq 1)$

Constructive Proof of Existence

If we want to prove:

- $\exists n \in \mathbf{Z}^{even}, \exists p,q, r,s \in \mathbf{Z}^{prime}\ n = p+q \wedge n = r+s \wedge p \neq r \wedge p \neq s \wedge q \neq r \wedge q \neq s$
  - let n=10
    - $n \in \mathbf{Z}^{even}$ by definition of even
  - Let p = 5 and the q = 5
    - $p,q \in \mathbf{Z}^{prime}$ by definition of prime
    - 10 = 5+5
  - Let r = 3 and s = 7
    - $r,s \in \mathbf{Z}^{prime}$ by definition of prime
    - 10 = 3+7
  - and all of the inequalities hold

Methods of Proving Universally Quantified Statements

- Method of Exhaustion
  - prove for each and every member of the domain
  - $\forall r \in \mathbf{Z}^+$ where $23<r<29 \rightarrow \exists\ p,q \in \mathbf{Z}^+ (r = p*q) \wedge (p<=q)$
- Generalizing from the "generic particular"
  - suppose x is a particular but arbitrarily chosen element of the domain
  - show that x satisfies the property
  - i.e. $\forall r \in \mathbf{Z},\ r \in Z^{even} \rightarrow r^2 \in Z^{even}$

Examples of Generalizing from the "Generic Particular"

- The product of any two odd integers is also odd.
  - $\forall m,n \in \mathbf{Z}, [(m \in \mathbf{Z^{odd}} \wedge n \in \mathbf{Z^{odd}}) \rightarrow m*n \in \mathbf{Z^{odd}}]$
- The product of any two rationals is also rational.
  - $\forall m,n \in \mathbf{Q}, m*n \in \mathbf{Q}$

Disproof by Counter Example

- $\forall\ r \in \mathbf{Z}, r^2 \in \mathbf{Z}^+ \rightarrow r \in \mathbf{Z}^+$
- Counter Example: $r^2 = 9$ ^ $r = -3$
  - $r^2 \in \mathbf{Z}^+$ since $9 \in \mathbf{Z}^+$ so the antecedent is true
  - but $r \notin \mathbf{Z}^+$ since $-3 \notin \mathbf{Z}^+$ so the consequent is false
  - this means the implication is false for $r = -3$ so this is a valid counter example
- When a counter example is given you must always justify that it is a valid counter example by showing the algebra (or other interpretation needed) to support your claim

Division definitions

- $d \mid n \leftrightarrow \exists k \in \mathbf{Z}, n = d*k$
- n is divisible by d
- n is a multiple of d
- d is a divisor of n
- d divides n

- standard factored form
  - $n = p_1^{e1} * p_2^{e2} * p_3^{e3} * \ldots * p_k^{ek}$

Proof using the Contrapositive

For all positive integers, if n does not divide a number to which d is a factor, then n can not divide d.

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$\forall n,d,c \in Z+,\ n \mid d \rightarrow n \mid dc$

proof:

more integer definitions

- div and mod operators
  - n div d --- integer quotient for $\frac{n}{d}$
  - n mod d --- integer remainder for $\frac{n}{d}$
  - (n div d = q) ^ (n mod d = r) $\leftrightarrow$ n = d*q+r

    where $n \in \mathbf{Z}^{\geq 0}$, $d \in \mathbf{Z}^{+}$, $r \in \mathbf{Z}$, $q \in \mathbf{Z}$, $0 \leq r < d$
- relating "mod" to "divides"
  - $d \mid n \quad \leftrightarrow \quad 0 = n \bmod d$

    $\leftrightarrow \quad 0 \equiv_d n$
- <u>definition of equivalence in a mod</u>
  - $x \equiv_d y \leftrightarrow d \mid (x-y)$ [note: their remainders are equal]
  - sometimes written as $x \equiv y \bmod d$ meaning $(x \equiv y) \bmod d$

Quotient Remainder Theorem

$\forall n \in Z\ \forall d \in Z^+\ \exists q,r \in Z$

$(n=dq+r)\ \wedge\ (0 \leq r < d)$

Proving definition of equiv in a mod by using the quotient remainder theorem

This means

prove that if $[m \equiv_d n]$, then $[d|(n-m)]$

where $m,n \in Z$ and $d \in Z^+$

Proofs using this definition

- $\forall m \in Z^+\ \forall a,b \in Z$

  $a \equiv_m b \leftrightarrow \exists\ k \in \mathbf{Z}\ \ a=b+km$

- $\forall m \in Z^+\ \forall a,b,c,d \in Z$

$a \equiv_m b\ \wedge\ c \equiv_m d \rightarrow a+c \equiv_m b+d$

Proof by Division into Cases

$\forall n \in Z\ 3 \not| n \rightarrow n^2 \equiv_3 1$

Floor and Ceiling Definitions

- n is the floor of x where $\mathbf{x \in R \wedge n \in Z}$

$$\lfloor x \rfloor = n \leftrightarrow n \leq x < n+1$$

- n is the ceiling of x where $\mathbf{x \in R \wedge n \in Z}$

$$\lceil x \rceil = n \leftrightarrow n-1 < x \leq n$$

Floor/Ceiling Proofs

- $\forall x,y \in R \lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

- $\forall x \in R \ \forall y \in Z \lfloor x+y \rfloor = \lfloor x \rfloor + y$

Proof by Division into Cases (again)

- The floor of (n/2) is either
  - a) n/2 when n is even
  - or b) (n-1)/2 when n is odd

Prime Factored Form

$n = p_1^{e1} * p_2^{e2} * p_3^{e3} * \ldots * p_k^{ek}$

- Unique Factorization Theorem (Theorem 3.3.3)
  - given any integer n>1
  - $\exists k \in Z, \exists p_1, p_2, \ldots p_k \in Z^{prime}, \exists e_1, e_2, \ldots e_k \in Z^{+},$

    where the p's are distinct and any other expression of n is identical to this except maybe in the order of the factors
- Standard Factored Form
  - $p_i < p_{i+1}$
- $\exists m \in Z, 8*7*6*5*4*3*2*m = 17*16*15*14*13*12*11*10$
  - Does $17 | m$??

Steps Toward Proving the Unique Factorization Theorem

- Every integer greater than or equal to 2 has at least one prime that divides it
- For all integers greater than 1,

  if $a | b$, then $a \not| (b+1)$
- There are an infinite number of primes

Using the Unique Factorization Theorem

- Prove that the $\sqrt{3} \notin Q$
- Prove:

$$\forall a \in Z^{+} \forall q \in Z^{prime} \quad q|a^2 \rightarrow \quad q|a$$

Summary of Proof Methods

- Constructive Proof of Existence
- Proof by Exhaustion
- Proof by Generalizing from the Generic Particular
- Proof by Contraposition
- Proof by Contradiction
- Proof by Division into Cases

Errors in Proofs

- Arguing from example for universal proof.
- Misuse of Variables
- Jumping to the Conclusion (missing steps)
- Begging the Question
- Using "if" about something that is known