

Sets

Definition of a Set:

NAME = {list of elements or description of elements}

i.e. $B = \{1,2,3\}$ or $C = \{x \in \mathbb{Z}^+ \mid -4 < x < 4\}$

Axiom of Extension:

A set is completely defined by its elements

i.e. $\{a,b\} = \{b,a\} = \{a,b,a\} = \{a,a,a,b,b,b\}$

Subset

$A \subseteq B \leftrightarrow \forall x \in U, x \in A \rightarrow x \in B$

A is contained in B

B contains A

$A \not\subseteq B \leftrightarrow \exists x \in U, x \in A \wedge x \notin B$

Relationship between membership and subset:

$\forall x \in U, x \in A \leftrightarrow \{x\} \subseteq A$

Definition of set equality: $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

Same Set or Not??

$$X = \{x \in \mathbb{Z} \mid \exists p \in \mathbb{Z}, x = 2p\}$$

$$Y = \{y \in \mathbb{Z} \mid \exists q \in \mathbb{Z}, y = 2q - 2\}$$

$$A = \{x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, x = 2i + 1\}$$

$$B = \{x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, x = 3i + 1\}$$

$$C = \{x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, x = 4i + 1\}$$

Set Operations

Formal Definitions and Venn Diagrams

Union:

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$

Intersection:

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$

Complement:

$$A^c = A' = \{x \in U \mid x \notin A\}$$

Difference:

$$A - B = \{x \in U \mid x \in A \wedge x \notin B\}$$

$$A - B = A \cap B'$$

Ordered n-tuple and the Cartesian Product

- Ordered n-tuple – takes order and multiplicity into account
- $(x_1, x_2, x_3, \dots, x_n)$
 - n values
 - not necessarily distinct
 - in the order given
- $(x_1, x_2, x_3, \dots, x_n) = (y_1, y_2, y_3, \dots, y_n) \leftrightarrow \forall i \in \mathbb{Z}^{1 \leq i \leq n}, x_i = y_i$

- Cartesian Product

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Formal Languages

- Σ = alphabet = a finite set of symbols
- string over Σ =
 - empty (or null) string denoted as ϵ
 - OR
 - ordered n-tuple of elements
- Σ^n = set of strings of length n
- Σ^* = set of all finite length strings

Empty Set Properties

1. $\emptyset$ is a subset of every set.
2. There is only one empty set.
3. The union of any set with $\emptyset$ is that set.
4. The intersection of any set with its own complement is $\emptyset$.
5. The intersection of any set with $\emptyset$ is $\emptyset$.
6. The Cartesian Product of any set with $\emptyset$ is $\emptyset$.
7. The complement of the universal set is $\emptyset$ and the complement of the empty set is the universal set.

Other Definitions

- Proper Subset

$$A \subset B \leftrightarrow A \subseteq B \wedge A \neq B$$

- Disjoint Set

A and B are disjoint

$\leftrightarrow$ A and B have no elements in common

$$\leftrightarrow \forall x \in U, x \in A \rightarrow x \notin B \wedge x \in B \rightarrow x \notin A$$

$A \cap B = \emptyset \leftrightarrow$ A and B are Disjoint Sets

- Power Set

$\mathcal{P}(A)$ = set of all subsets of A

Properties of Sets in Theorems 5.2.1 & 5.2.2

- Inclusion $A \cap B \subseteq A$ $A \cap B \subseteq B$
 $A \subseteq A \cup B$ $B \subseteq A \cup B$
- Transitivity
 $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
- DeMorgan's for Complement
 $(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
- Distribution of union and intersection
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Using Venn Diagrams to help
find counter example

$$A \cup (B \cap C) = ? = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = ? = (A \cap B) \cap C$$

Deriving new Properties
using rules and Venn diagrams

$$B - (A \cap C) = (B - A) \cup (B - C)$$

$$A - B = A - (A \cap B)$$

$$A \subseteq B \wedge A \subseteq C \rightarrow A \subseteq (B \cap C)$$

Partitions of a set

- A collection of nonempty sets
 $\{A_1, A_2, \dots, A_n\}$ is a partition of the set A
if and only if
 1. $A = A_1 \cup A_2 \cup \dots \cup A_n$
 2. $A_1, A_2, \dots, A_n$ are mutually disjoint

Proofs about Power Sets

Power set of $A = \mathcal{P}(A)$ = Set of all subsets of A

- Prove that

$$\forall A, B \in \{\text{sets}\}, A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$$

- Prove that (where $n(X)$ means the size of set X)

$$\forall A \in \{\text{sets}\}, n(A) = k \rightarrow n(\mathcal{P}(A)) = 2^k$$