Sets

Definition of a Set:

NAME = {list of elements or description of elements}

i.e. $B = \{1,2,3\}$ or $C = \{x \in Z^+ \mid -4 < x < 4\}$

Axiom of Extension:

A set is completely defined by its elements

i.e. {a,b} = {b,a} = {a,b,a} = {a,a,a,b,b,b}

Subset

$A \subseteq B \leftrightarrow \forall x \in U,\ x \in A \rightarrow x \in B$

A is contained in B

B contains A

$A \not\subseteq B \leftrightarrow \exists x \in U,\ x \in A \ \wedge\ x \notin B$

Relationship between membership and subset:

$\forall x \in U,\ x \in A \leftrightarrow \{x\} \subseteq A$

Definition of set equality: $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

Same Set or Not??

$X=\{x \in Z \mid \exists p \in Z,\ x = 2p\}$

$Y=\{y \in Z \mid \exists q \in Z,\ y = 2q-2\}$

$A=\{x \in Z \mid \exists i \in Z,\ x = 2i+1\}$

$B=\{x \in Z \mid \exists i \in Z,\ x = 3i+1\}$

$C=\{x \in Z \mid \exists i \in Z,\ x = 4i+1\}$

Set Operations

Formal Definitions and Venn Diagrams

Union:

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$

Intersection:

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$

Complement:

$$A^c = A' = \{x \in U \mid x \notin A\}$$

Difference:

$$A - B = \{x \in U \mid x \in A \wedge x \notin B\}$$

$$A - B = A \cap B'$$

Ordered n-tuple and the Cartesian Product

- Ordered n-tuple – takes order and multiplicity into account
- $(x_1,x_2,x_3,\ldots,x_n)$
  - n values
  - not necessarily distinct
  - in the order given
- $(x_1,x_2,x_3,\ldots,x_n) = (y_1,y_2,y_3,\ldots,y_n) \leftrightarrow \forall i \in Z^{1\le i\le n},\ x_i=y_i$

- Cartesian Product

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Formal Languages

- Σ = alphabet = a finite set of symbols
- string over Σ =

  empty (or null) string denoted as ε

  OR

  ordered n-tuple of elements
- Σ^n = set of strings of length n
- Σ^* = set of all finite length strings

Empty Set Properties

1. Ø is a subset of every set.
2. There is only one empty set.
3. The union of any set with Ø is that set.
4. The intersection of any set with its own complement is Ø.
5. The intersection of any set with Ø is Ø.
6. The Cartesian Product of any set with Ø is Ø.
7. The complement of the universal set is Ø and the complement of the empty set is the universal set.

Other Definitions

- Proper Subset

$$A \subset B \leftrightarrow A \subseteq B \wedge A \neq B$$

- Disjoint Set

A and B are disjoint

$\leftrightarrow$A and B have no elements in common

$\leftrightarrow \forall x \in U,\ x \in A \rightarrow x \notin B \ \wedge\ x \in B \rightarrow x \notin A$

$A \cap B = \emptyset \leftrightarrow$ A and B are Disjoint Sets

- Power Set

$\mathcal{P}(A)$ = set of all subsets of A

Properties of Sets in Theorems 5.2.1 & 5.2.2

- Inclusion $A \cap B \subseteq A$ $A \cap B \subseteq B$
  $A \subseteq A \cup B$ $B \subseteq A \cup B$
- Transitivity
  $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
- DeMorgan's for Complement
  $(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
- Distribution of union and intersection
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Using Venn Diagrams to help find counter example

$$A \cup (B \cap C) = ? = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = ? = (A \cap B) \cap C$$

Deriving new Properties
using rules and Venn diagrams

$$B - (A \bigcap C) = (B - A) \bigcup (B - C)$$

$$A - B = A - (A \bigcap B)$$

$$A \subseteq B \wedge A \subseteq C \rightarrow A \subseteq (B \bigcap C)$$

Partitions of a set

- A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of the set A

if and only if

1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$
2. $A_1, A_2, \ldots, A_n$ are mutually disjoint

Proofs about Power Sets

Power set of A = $\mathcal{P}$(A) = Set of all subsets of A

- Prove that

  $\forall$**A,B** $\in$ **{sets}, A**$\subseteq$**B** $\rightarrow$ $\mathcal{P}$**(A)** $\subseteq$ $\mathcal{P}$**(B)**

- Prove that (where n(X) means the size of set X)

  $\forall$**A** $\in$ **{sets}, n(A) = k** $\rightarrow$ **n(**$\mathcal{P}$**(A)) =** $\mathbf{2^k}$