CMSC 132: Object-Oriented Programming II

Algorithmic Complexity I

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Algorithm Efficiency

- Efficiency
  - Amount of resources used by algorithm
    - Time, space
- Measuring efficiency
  - Benchmarking
  - Asymptotic analysis

Benchmarking

- Approach
  - Pick some desired inputs
  - Actually run implementation of algorithm
  - Measure time & space needed
- Industry benchmarks
  - SPEC – CPU performance
  - MySQL – Database applications
  - WinStone – Windows PC applications
  - MediaBench – Multimedia applications
  - Linpack – Numerical scientific applications

Benchmarking

- Advantages
  - Precise information for given configuration
    - Implementation, hardware, inputs
- Disadvantages
  - Affected by configuration
    - Data sets (often too small)
      - a dataset that was the right size 3 years ago is likely too small now
    - Hardware
    - Software
  - Affected by special cases (biased inputs)
  - Does not measure intrinsic efficiency

Asymptotic Analysis

- **Approach**
  - **Mathematically analyze efficiency**
  - **Calculate time as function of input size n**
    - **T ≈ O(f(n))**
    - **T is on the order of f(n)**
    - **“Big O” notation**
- **Advantages**
  - **Measures intrinsic efficiency**
  - **Dominates efficiency for large input sizes**
  - **Programming language, compiler, processor irrelevant**

Search Example

- **Number guessing game**
  - **Pick a number between 1…n**
  - **Guess a number**
  - **Answer “correct”, “too high”, “too low”**
  - **Repeat guesses until correct number guessed**

Linear Search Algorithm

- Algorithm
  - Guess number = 1
  - If incorrect, increment guess by 1
  - Repeat until correct
- Example
  - Given number between 1…100
  - Pick 20
  - Guess sequence = 1, 2, 3, 4 … 20
  - Required 20 guesses

Linear Search Algorithm

- Analysis of # of guesses needed for 1…n
  - If number = 1, requires 1 guess
  - If number = n, requires n guesses
  - On average, needs n/2 guesses
  - Time = O(n) = Linear time

Binary Search Algorithm

- **Algorithm**
  - **Set low and high to be lowest and highest possible value**
  - **Guess middle = (low+high)/2**
  - **If too large, set high = middle-1**
  - **If too small, set low = middle+1**
  - **Repeat until guess correct**

Binary Search Algorithm

- Example
  - Given number between 1…100
  - secret number we are trying is find is 20
  - Guesses
    - low = 1, high = 100, guess 50, Answer = too large
    - low = 1, high = 49, guess 25, Answer = too large
    - low = 1, high = 24, guess 12, Answer = too small
    - low = 13, high = 24, guess 18, Answer = too small
    - low = 19, high = 24, guess 21, Answer = too large
    - low = 19, high = 20, guess 19, Answer = too small
    - low = 20, high = 20, guess 20, Answer = correct
    - Required 7 guesses

Binary Search Algorithm

- **Analysis of # of guesses needed for 1…n**
  - **If number = n/2, requires 1 guess**
  - **If number = 1, requires $\log_2(n)$ guesses**
  - **If number = n, requires $\log_2(n)$ guesses**
  - **On average, needs $\log_2(n)$ guesses**
  - **Time = $O(\log_2(n)) = O(\log(n))$ = Log time**

Search Comparison

- **For number between 1…100**
  - **Simple algorithm = 50 steps**
  - **Binary search algorithm = $\log_2(n)$ = 7 steps**
- **For number between 1…100,000**
  - **Simple algorithm = 50,000 steps**
  - **Binary search algorithm = $\log_2(n)$ (about 17 steps)**
- **Binary search is much more efficient!**

Asymptotic Complexity

- **Comparing two linear functions**

Size	Running Time	
	n/2	4n+3
64	32	259
128	64	515
256	128	1027
512	256	2051

Asymptotic Complexity

- **Comparing two functions**
  - **n/2 and 4n+3 behave similarly**
  - **Run time roughly doubles as input size doubles**
  - **Run time increases linearly with input size**
- **For large values of n**
  - **Time(2n) / Time(n) approaches exactly 2**
- **Both are O(n) programs**

Asymptotic Complexity

- **Comparing two log functions**

Size	Running Time	
	$\log_2(n)$	$5 * \log_2(n) + 3$
64	6	33
128	7	38
256	8	43
512	9	48

Asymptotic Complexity

- **Comparing two functions**
  - **$\log_2(n)$ and $5 * \log_2(n) + 3$ behave similarly**
  - **Run time roughly increases by constant as input size doubles**
  - **Run time increases logarithmically with input size**
- **For large values of n**
  - **Time(2n) – Time(n) approaches constant**
  - **Base of logarithm does not matter**
    - **Simply a multiplicative factor**

$$\log_a N = (\log_b N) / (\log_b a)$$

- **Both are O(log(n)) programs**

Asymptotic Complexity

- **Comparing two quadratic functions**

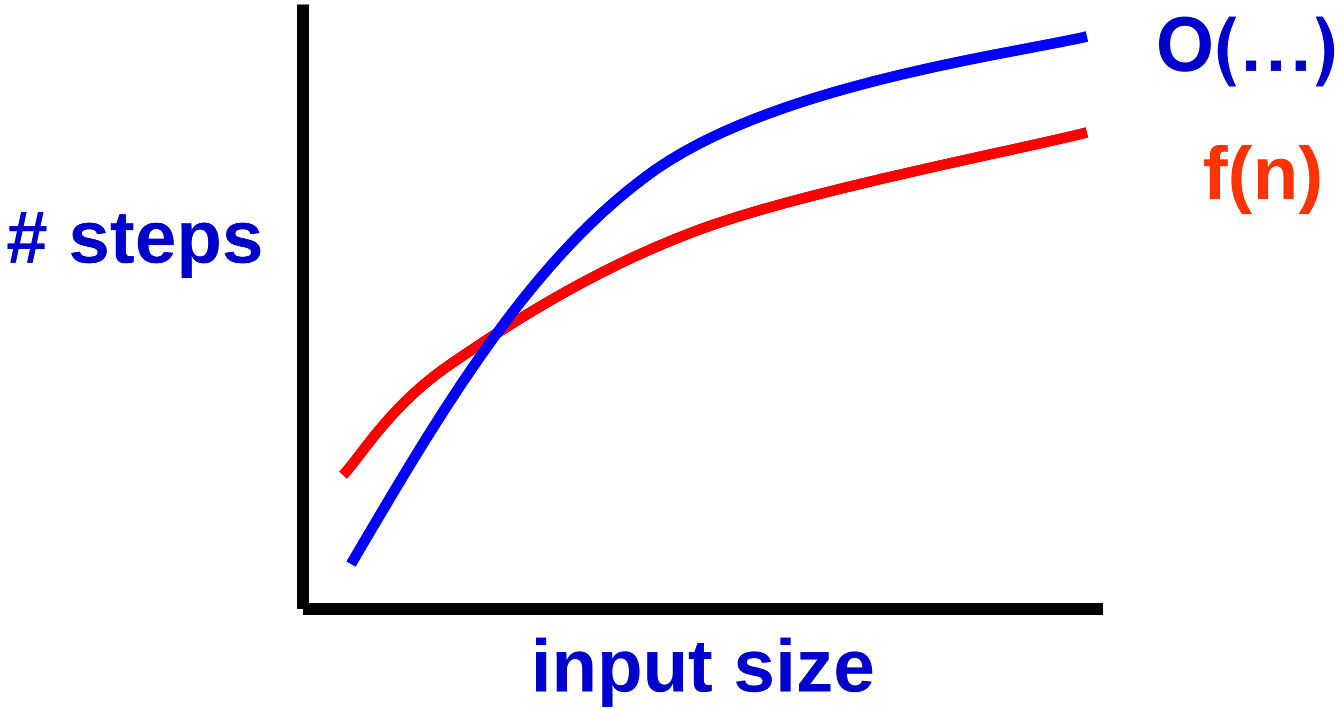
Size	Running Time	
	n^2	$2\,n^2 + 8$
2	4	16
4	16	40
8	64	132
16	256	520

Asymptotic Complexity

- Comparing two functions
  - n^2 and $2\ n^2 + 8$ behave similarly
  - Run time roughly increases by 4 as input size doubles
  - Run time increases quadratically with input size
- For large values of n
  - Time(2n) / Time(n) approaches 4
- Both are O(n^2) programs

Big-O Notation

- Represents
  - Upper bound on number of steps in algorithm
    - For sufficiently large input size
  - Intrinsic efficiency of algorithm for large inputs



Formal Definition of Big-O

- Function f(n) is O(g(n)) if
  - For some positive constants M, N_0
  - $M \times g(n) \geq f(n)$, for all $n \geq N_0$
- Intuitively
  - For some coefficient M & all data sizes $\geq N_0$
    - $M \times g(n)$ is always greater than f(n)

Big-O Examples

- $5n + 1000 \Rightarrow O(n)$
  - Select $M = 6$, $N_0 = 1000$
  - For $n \geq 1000$
    - $6n \geq 5n+1000$ is always true
  - Example $\Rightarrow$ for $n = 1000$
    - $6000 \geq 5000 + 1000$

Big-O Examples

- $2n^2 + 10n + 1000 \Rightarrow O(n^2)$
  - Select $M = 4$, $N_0 = 100$
  - For $n \geq 100$
    - $4n^2 \geq 2n^2 + 10n + 1000$ is always true
  - Example $\Rightarrow$ for $n = 100$
    - $40000 \geq 20000 + 1000 + 1000$

Observations

- **For large values of n**
  - **Any O(log(n)) algorithm is faster than O(n)**
  - **Any O(n) algorithm is faster than O(n^2)**
- **Asymptotic complexity is fundamental measure of efficiency**

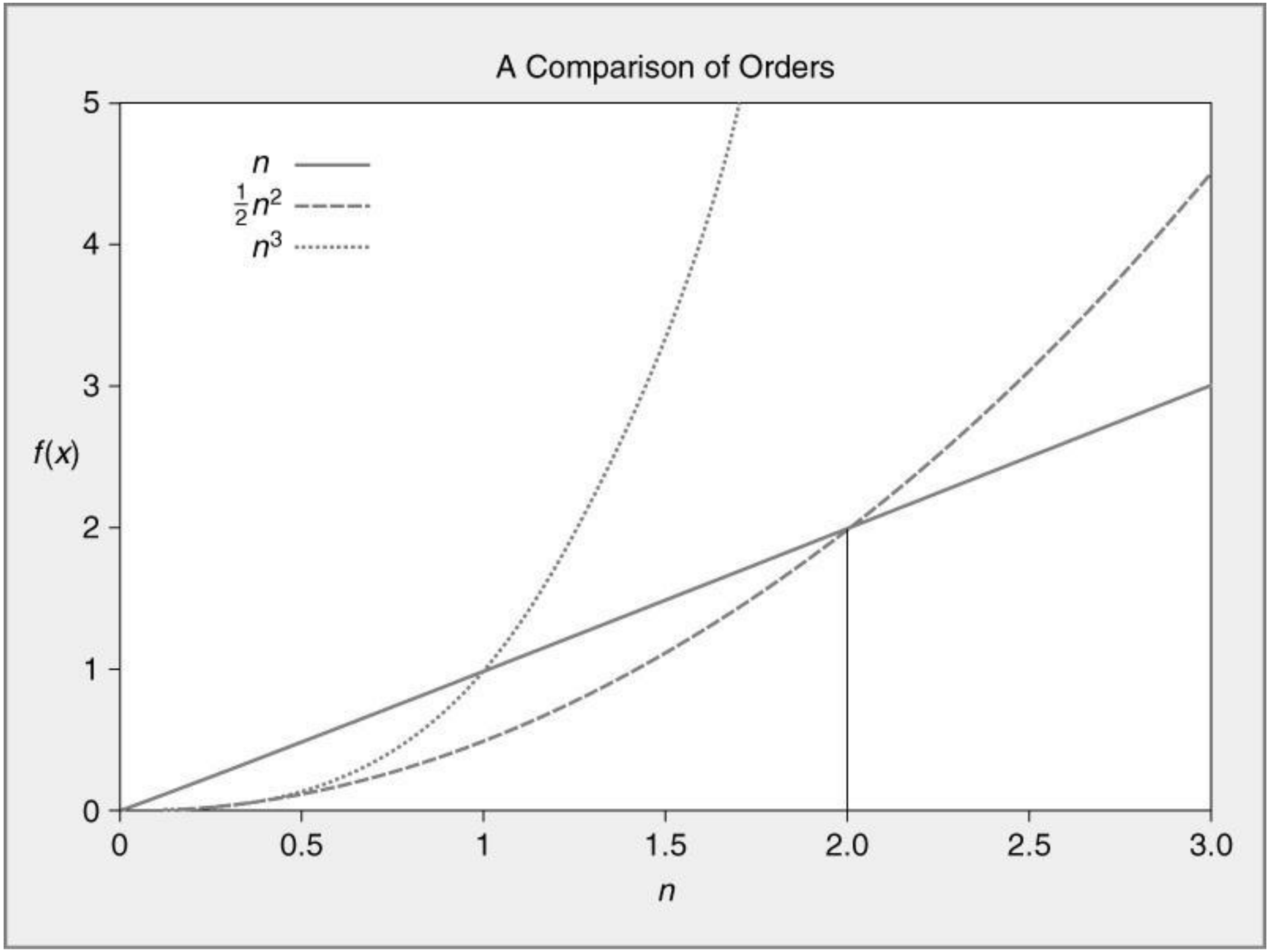
Asymptotic Complexity Categories

Complexity	Name	Example
$O(1)$	Constant	Array access
$O(\log(n))$	Logarithmic	Binary search
$O(n)$	Linear	Largest element
$O(n \log(n))$	N log N	Optimal sort
$O(n^2)$	Quadratic	2D Matrix addition
$O(n^3)$	Cubic	2D Matrix multiply
$O(n^k)$	Polynomial	Linear programming
$O(k^n)$	Exponential	Integer programming
$O(n!)$	Factorial	Brute-force search TSP
$O(n^n)$	N to the N	

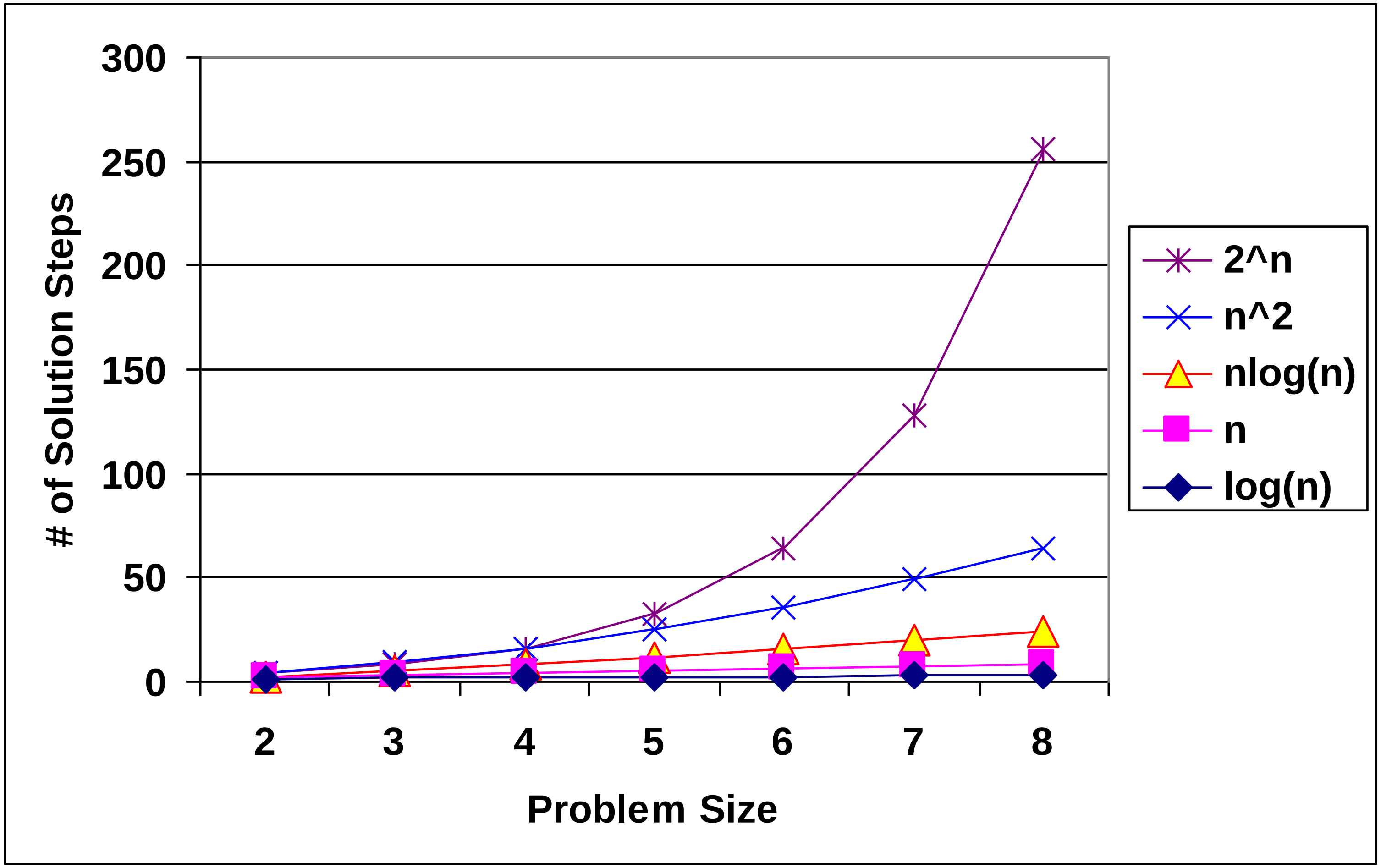
From smallest to largest

For size n, constant $k > 1$

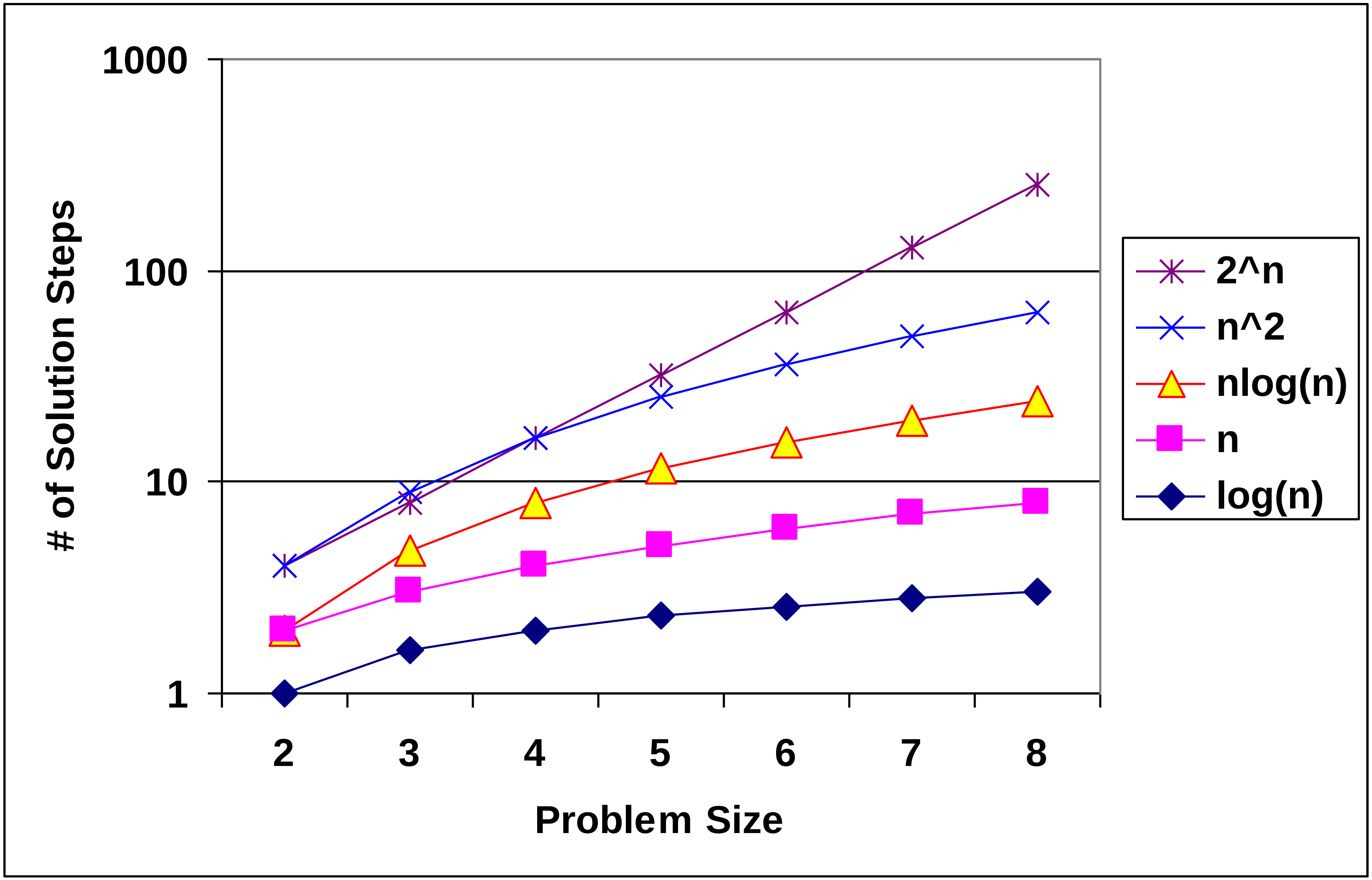
Comparison of Complexity



Complexity Category Example



Complexity Category Example

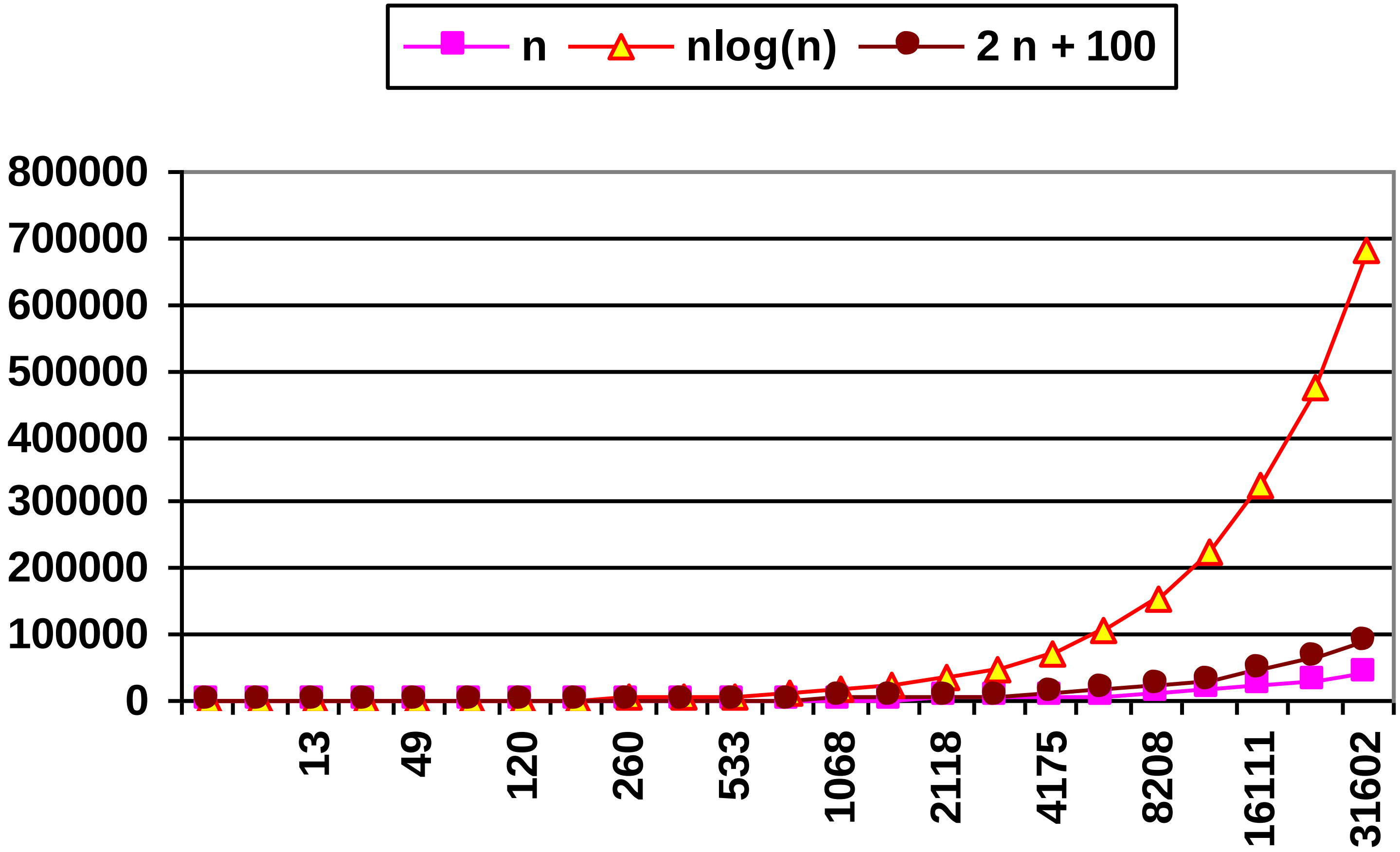


Calculating Asymptotic Complexity

- As n increases
  - Highest complexity term dominates
  - Can ignore lower complexity terms
- Examples
  - $2\,n + 100 \Rightarrow O(n)$
  - $n \log(n) + 10\,n \Rightarrow O(n\log(n))$
  - $\frac{1}{2}\,n^2 + 100\,n \Rightarrow O(n^2)$
  - $n^3 + 100\,n^2 \Rightarrow O(n^3)$
  - $1/100\;2^n + 100\,n^4 \Rightarrow O(2^n)$

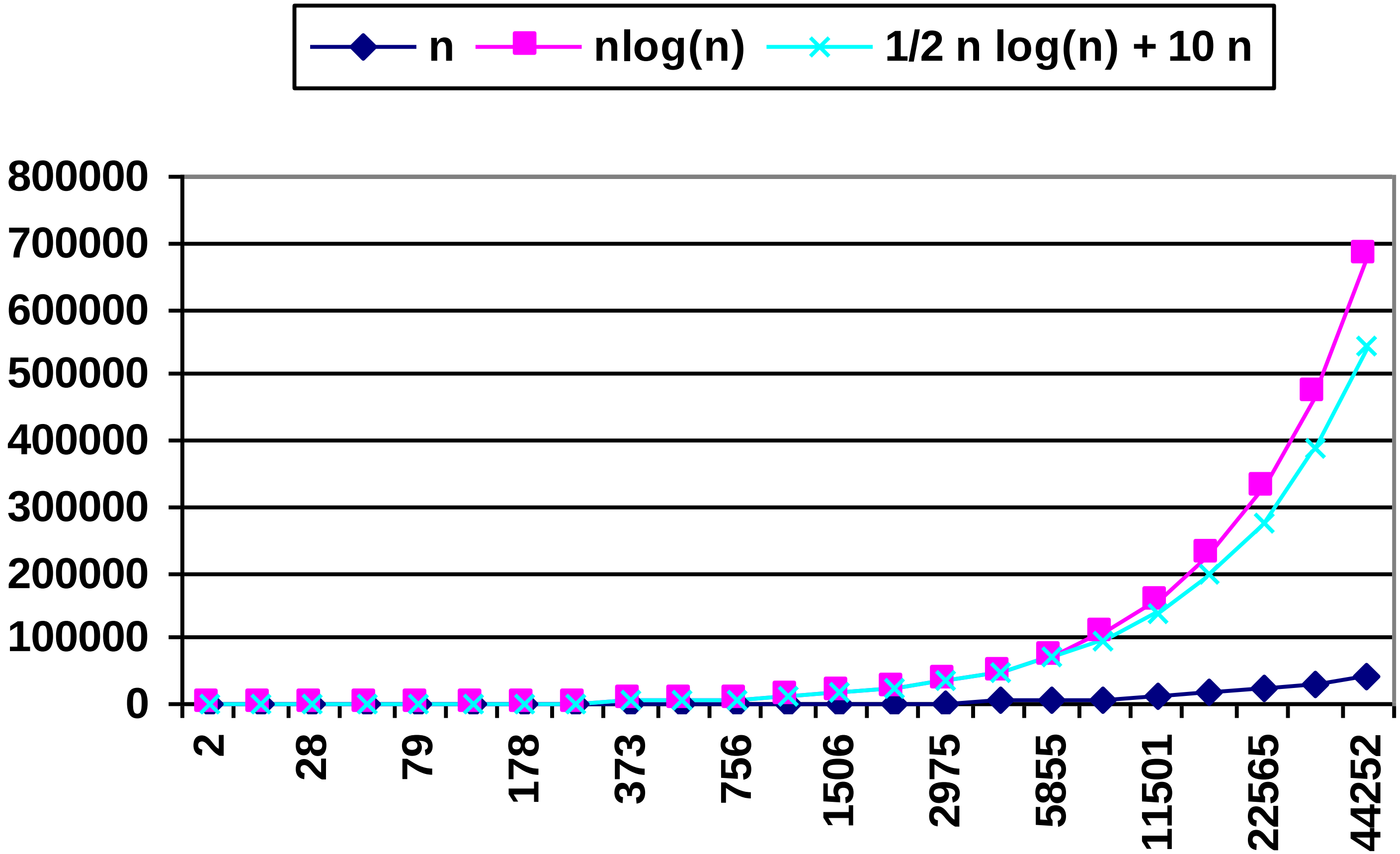
Complexity Examples

- $2n + 100 \Rightarrow O(n)$



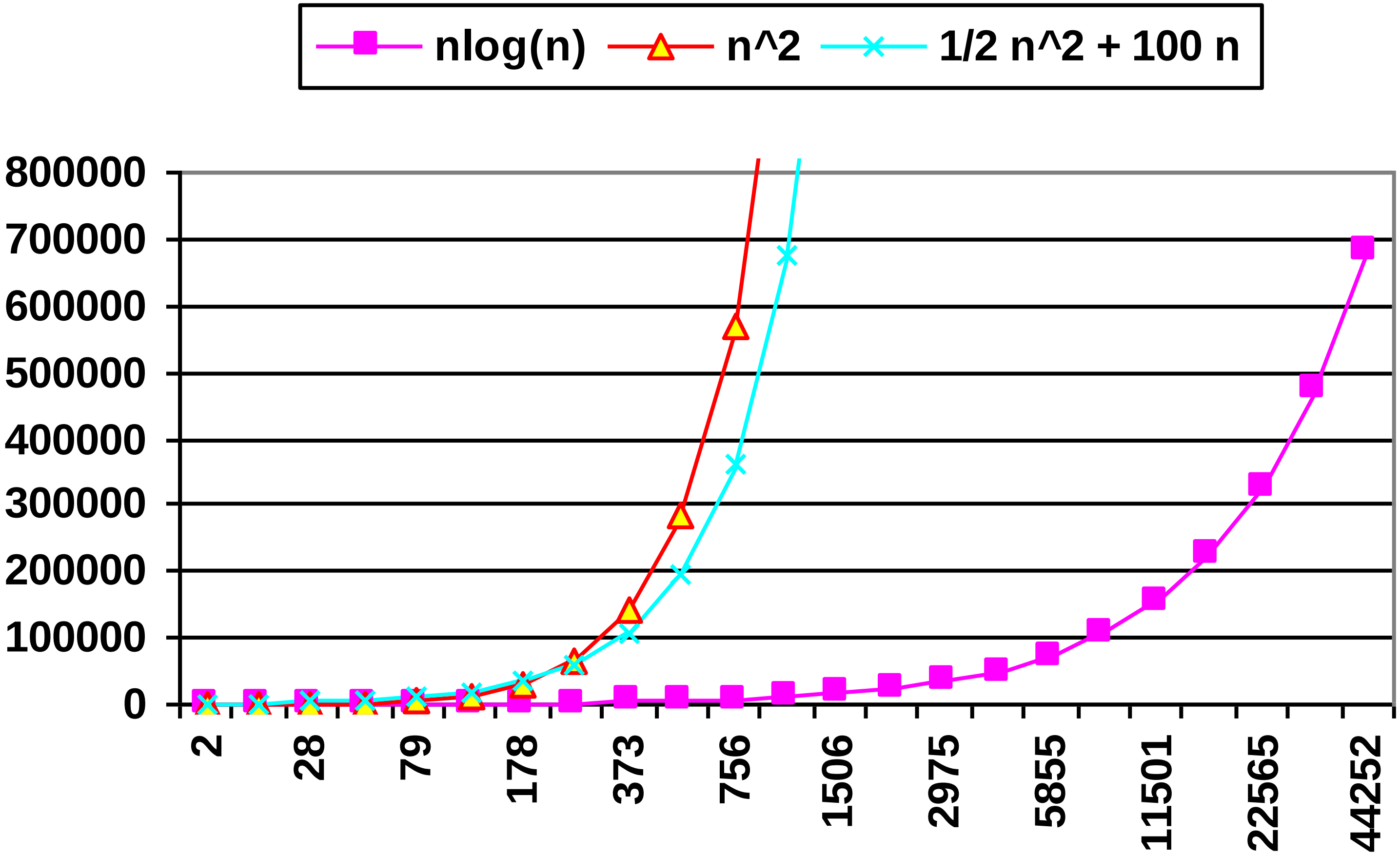
Complexity Examples

- $\frac{1}{2}$ n log(n) + 10 n $\Rightarrow$ O(nlog(n))



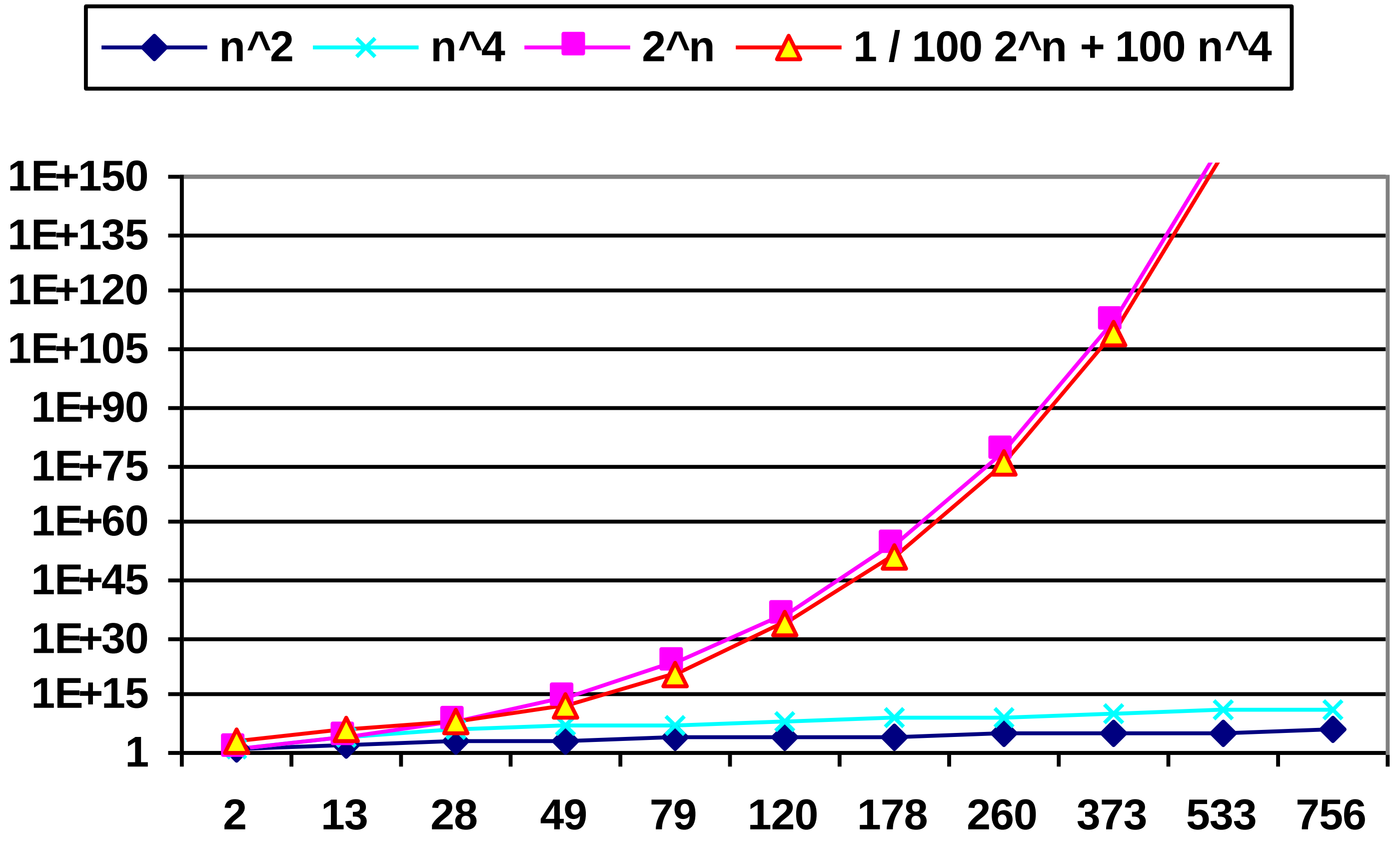
Complexity Examples

- $\frac{1}{2} n^2 + 100\, n \Rightarrow O(n^2)$



Complexity Examples

- $1/100\ 2^n + 100\ n^4 \Rightarrow O(2^n)$



Types of Case Analysis

- Can analyze different types (cases) of algorithm behavior
- Types of analysis
  - Best case
  - Worst case
  - Average case
  - Amortized

Types of Case Analysis

- **Best case**
  - **Smallest number of steps required**
  - **Not very useful**
  - **Example $\Rightarrow$ Find item in first place checked**

Types of Case Analysis

- **Worst case**
  - **Largest number of steps required**
  - **Useful for upper bound on worst performance**
    - **Real-time applications (e.g., multimedia)**
    - **Quality of service guarantee**
  - **Example $\Rightarrow$ Find item in last place checked**

Quicksort Example

- Quicksort
  - One of the fastest comparison sorts
  - Frequently used in practice
- Quicksort algorithm
  - Pick pivot value from list
  - Partition list into values smaller & bigger than pivot
  - Recursively sort both lists

Quicksort Example

- Quicksort properties
  - Average case = O(nlog(n))
  - Worst case = $O(n^2)$
    - Pivot ≈ smallest / largest value in list
    - Picking from front of nearly sorted list
- Can avoid worst-case behavior
  - Select random pivot value

Types of Case Analysis

- **Average case**
  - **Number of steps required for “typical” case**
  - **Most useful metric in practice**
  - **Different approaches**
    - **Average case**
    - **Expected case**

Approaches to Average Case

- **Average case**
  - **Average over all possible inputs**
    - **Assumes all inputs have the same probability**
  - **Example**
    - **Case 1 = 10 steps, Case 2 = 20 steps**
    - **Average = 15 steps**
- **Expected case**
  - **Weighted average over all possible inputs**
    - **Based on probability of each input**
  - **Example**
    - **Case 1 (90%) = 10 steps, Case 2 (10%) = 20 steps**
    - **Average = 11 steps**

Average Case Example

- Example problem
  - Average # of comparisons needed to find a number in the (sorted) array A[ ] = {1, 4, 8, 12, 15} using
    - Linear search
      - Start from beginning, compare elements one at a time
    - Binary search
      - Start from middle of array at index k, compare element
      - If not element, repeat for top or bottom half of remaining array depending on whether element is smaller or greater than A[k]

Average Case : Linear Search

- **Algorithm**
  - Find # of comparisons needed for each case
    - 1 → 1 comparison (1)
    - 4 → 2 comparisons (1, 4)
    - 8 → 3 comparisons (1, 4, 8)
    - 12 → 4 comparisons (1, 4, 8, 12)
    - 15 → 5 comparisons (1, 4, 8, 12, 15)
  - Calc average = total # of comparisons / # cases
    - Total # comparisons = 1 + 2 + 3 + 4 + 5 = 15
    - # cases = 5
    - Average = 3 comparisons / number

Average Case : Binary Search

- **Algorithm**
  - **Find # of comparisons needed for each case**
    - **1 $\rightarrow$ 3 comparisons (8, 4, 1)**
    - **4 $\rightarrow$ 2 comparisons (8, 4)**
    - **8 $\rightarrow$ 1 comparisons (8)**
    - **12 $\rightarrow$ 2 comparisons (8, 12)**
    - **15 $\rightarrow$ 3 comparisons (8, 12, 15)**
  - **Calc average = total # of comparisons / # cases**
    - **Total # comparisons = 3 + 2 + 1 + 2 + 3 = 11**
    - **# cases = 5**
    - **Average = 2.2 comparisons / number**

Average Case Example

- **Example problem 2**
  - **Average # of comparisons needed to find a number in a sorted array A[n] of size n using**
    - **Linear search**
    - **Binary search**
  - **For simplicity, we assume elements are stored in A[1] … A[n]**

Average Case : Linear Search

- Algorithm
  - Find # of comparisons needed for each case
    - A[1] → 1 comparison (A[1])
    - A[2] → 2 comparisons (A[1], A[2])
    - …
    - A[n] → n comparisons (A[1] … A[n])
  - Calc average = total # of comparisons / # cases
    - Total # comparisons = 1 + 2 + … + n = ½ n2 + 1
    - # cases = n
    - Average ≈ ½ n comparisons / number

Average Case : Binary Search

- Algorithm
  - Find # of comparisons needed for each case
    - A[n/2] → 1 comp (A[n/2])
    - A[n/4], A[3n/4] → 2 comps (A[n/2], A[n/4])
    - …
    - A[1], A[3]…A[n] → log2(n) comparisons
    - (A[n/2], A[n/4], A[n/8]…A[1])
  - Calc average = total # of comparisons / # cases
    - Total # comparisons = n/2 * log2(n) +
    - n/4 * log2(n)–1 + … + 1 = n log2(n)
    - # cases = n
    - Average ≈ log2(n) comparisons / number

Sample problem

- Given an array a of integers
  - find the subrange that has the maximum sum
    - e.g., find low, high that maximizes
      a[low] + a[low+1] + ... + a[high]
    - only non empty ranges (low <= high)
    - If a contained only nonnegative integers, would be low = 0, high = a.length -1
    - but a can contain negative numbers
    - Can assume that arithmetetic overflow isn't an issue

One solution

```
public static int findBestRange(int [] a) {
int bestSum = a[0];
for(int low = 0; low < a.length; low++)
  for(int high = low; high < a.length; high++) {
    int sum = 0;
    for(int i = low; i <= high; i++) sum += a[i];
    if (bestSum < sum)
      bestSum = sum;
     }
return bestSum;
}

// What is the complexity of the algorithm used here?
```

Can you find a better algorithm?