# CMSC 132: Object-Oriented Programming II

## Minimal Spanning Tree Algorithms

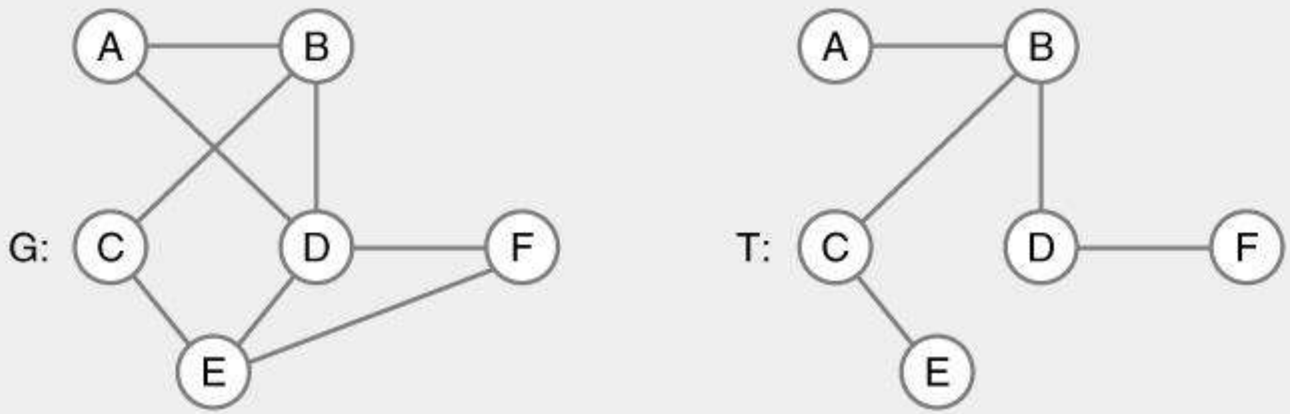
Department of Computer Science
University of Maryland, College Park

# Overview

- Spanning trees
- Minimum spanning tree (MST)
  - Prim’s algorithm
  - Kruskal’s algorithm
- Graph implementation
  - Adjacency list / matrix / set

# Spanning Tree

- **Set of edges connecting all nodes in graph**
  - **need N-1 edges for N nodes**
  - **no cycles, can be thought of as a tree**
- **Can build tree during traversal**




(a) Graph G

(b) Spanning tree T of graph G

# Spanning Tree Construction

- **Recursive algorithm**

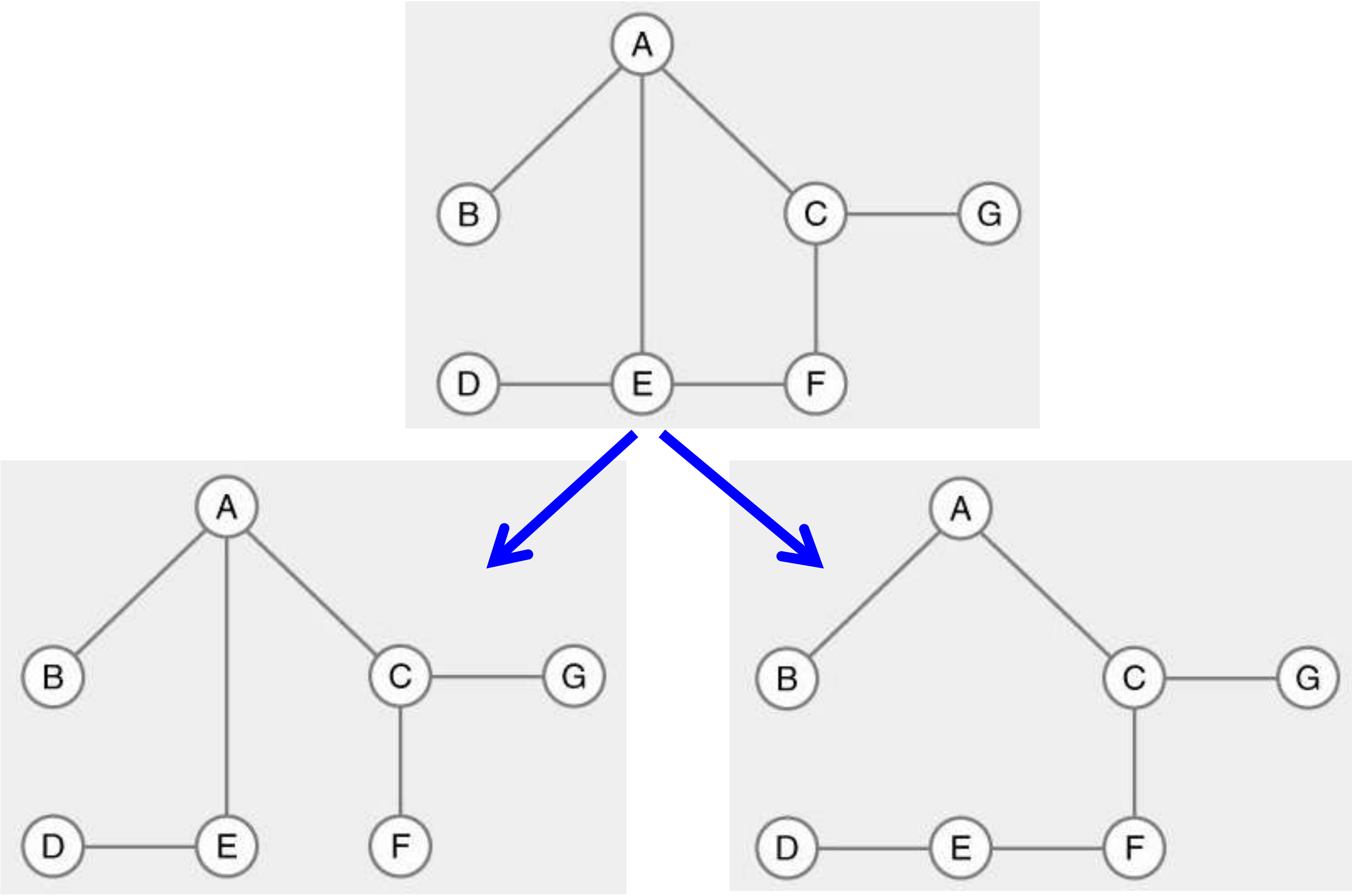
```
Known = { start }
explore ( start );

void explore (Node X) {
    for each successor Y of X
        if (Y is not in Known)
            Parent[Y] = X
            Add Y to Known
            explore(Y)
}
```

# Spanning Tree Construction

- **Iterative algorithm**

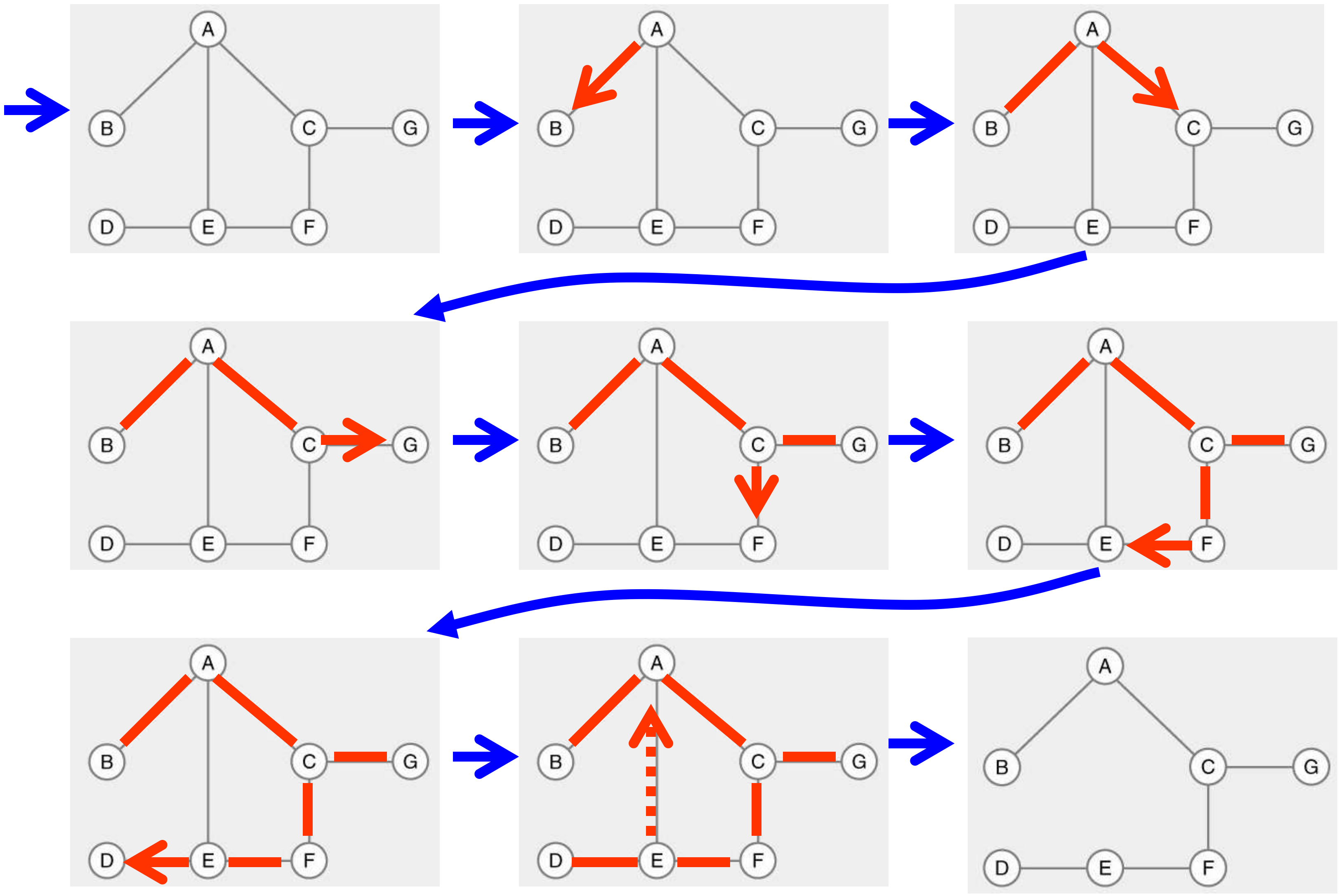
```
Known = { start }
Discovered = { start }
while ( Discovered ≠ ∅ ) {
    take node X out of Discovered
    for each successor Y of X
        if (Y is not in Known)
            Parent[Y] = X
            Add Y to Discovered
            Add Y to Known
}
```

# Breadth & Depth First Spanning Trees

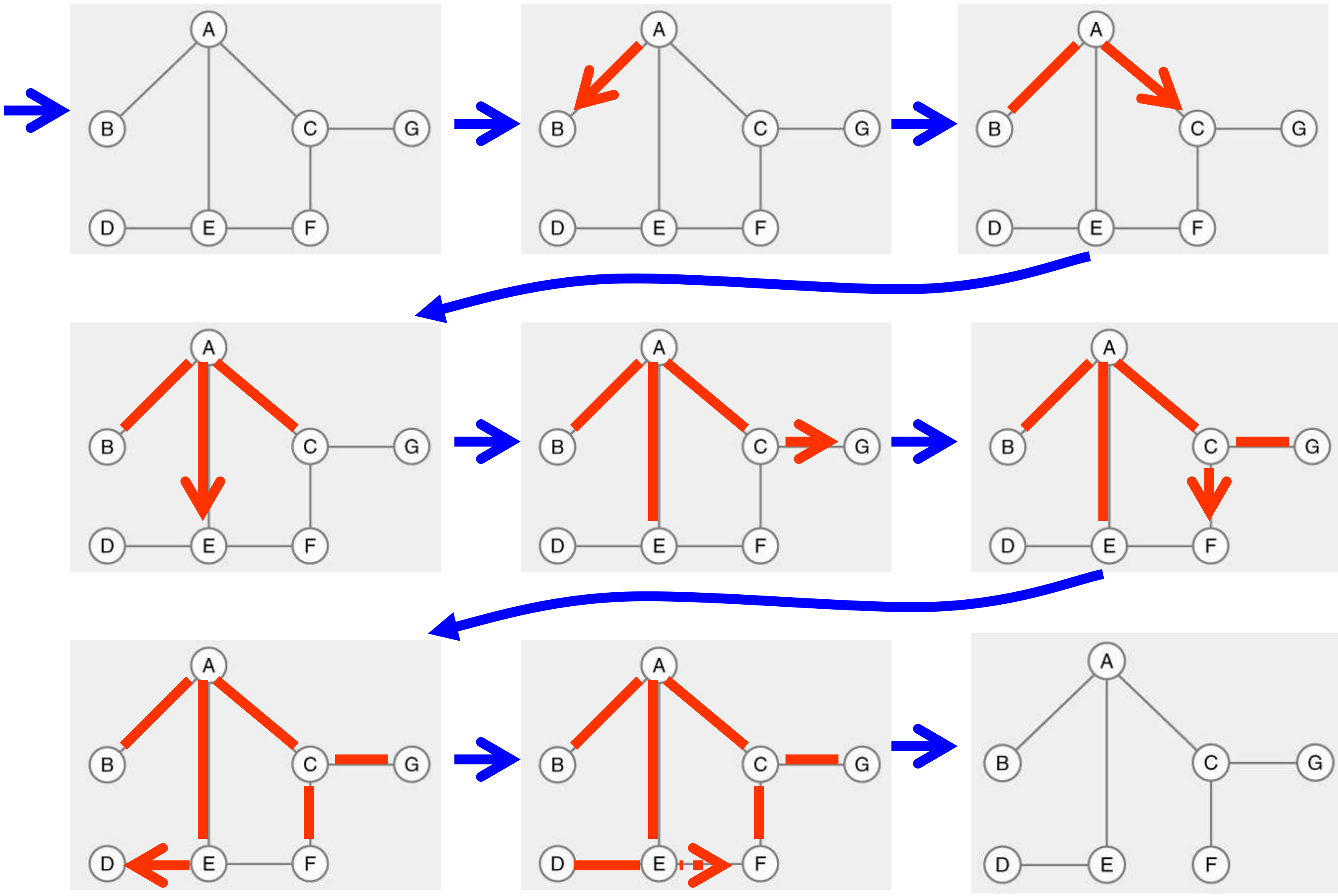



**Breadth-first** **Depth-first**

# Depth-First Spanning Tree Example



# Breadth-First Spanning Tree Example

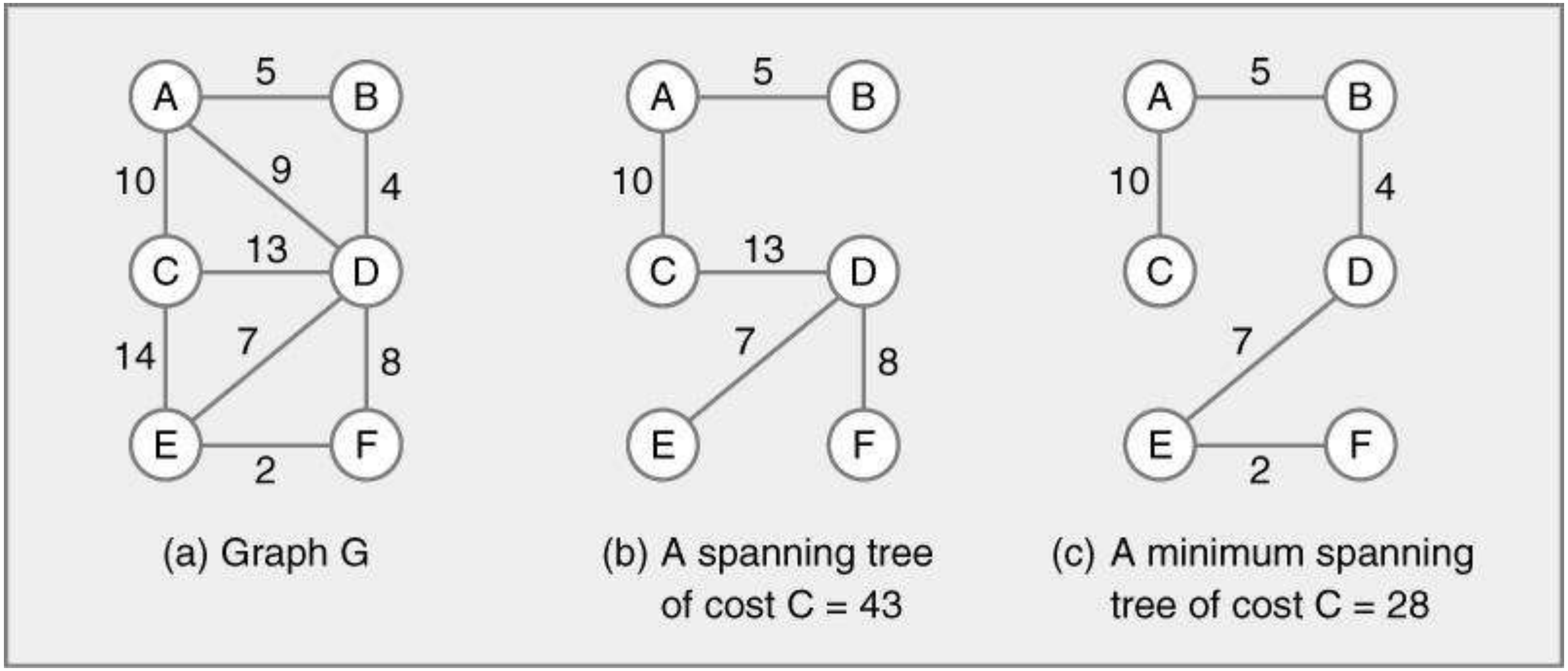


# Spanning Tree Construction

- Many spanning trees possible
  - Different breadth-first traversals
    - Nodes same distance visited in different order
  - Different depth-first traversals
    - Neighbors of node visited in different order
  - Different traversals yield different spanning trees

# Minimum Spanning Tree (MST)

- **Spanning tree with minimum total edge weight**



# Minimum Spanning Tree (MST)

- **Possible to have multiple MSTs**
  - **Different spanning trees with same weight**

- **Example applications**
  - **Minimize length of telephone lines for neighborhood**
  - **Minimize distance of airplane routes serving cities**

# Algorithms for Finding MST

- **Three well known algorithms**

1. **Borůvka's algorithm** [1926]
   - For constructing efficient electricity network
   - Rediscovered by Sollin in 1960s
2. **Prim's algorithm** [1957]
   - First discovered by Vojtěch Jarník in 1930
   - Similar to Djikstra's algorithm
3. **Kruskal's algorithm** [1956]
   - By Prof. Clyde Kruskal's uncle

# Algorithms for Finding MST

1. **Borůvka's algorithm**
   - Add vertices to MST in parallel
2. **Prim's algorithm**
   - Add vertices to MST
     - One at a time
     - Closest vertex first
3. **Kruskal's algorithm**
   - Add edges to MST
     - One at a time
     - Lightest edge first

# Shortest Path – Dijkstra's Algorithm

```
S = ∅
P[ ] = none for all nodes
C[start] = 0, C[ ] = ∞ for all other nodes
while ( not all nodes in S )
    find node K not in S with smallest C[K]
    add K to S
    for each node J not in S  adjacent to K
        if ( C[K] + cost of (K,J) < C[J] )
            C[J] = C[K] + cost of (K,J)
            P[J] = K
```

Optimal solution computed with greedy algorithm

# MST – Prim's Algorithm

```
S = ∅
P[ ] = none for all nodes
C[start] = 0, C[ ] = ∞ for all other nodes
while ( not all nodes in S )
    find node K not in S with smallest C[K]
    add K to S
    for each node J not in S  adjacent to K
        if ( /* C[K] + */ cost of (K,J) < C[J] )
            C[J] = /* C[K] + */ cost of (K,J)
            P[J] = K
```

Keeps track of vertex w/ minimal distance to current tree

Optimal solution computed with greedy algorithm

# MST – Kruskal's Algorithm

sort edges by weight (from least to most)

tree = $\varnothing$

for each edge (X,Y) in order

    if it does not create a cycle

        add (X,Y) to tree

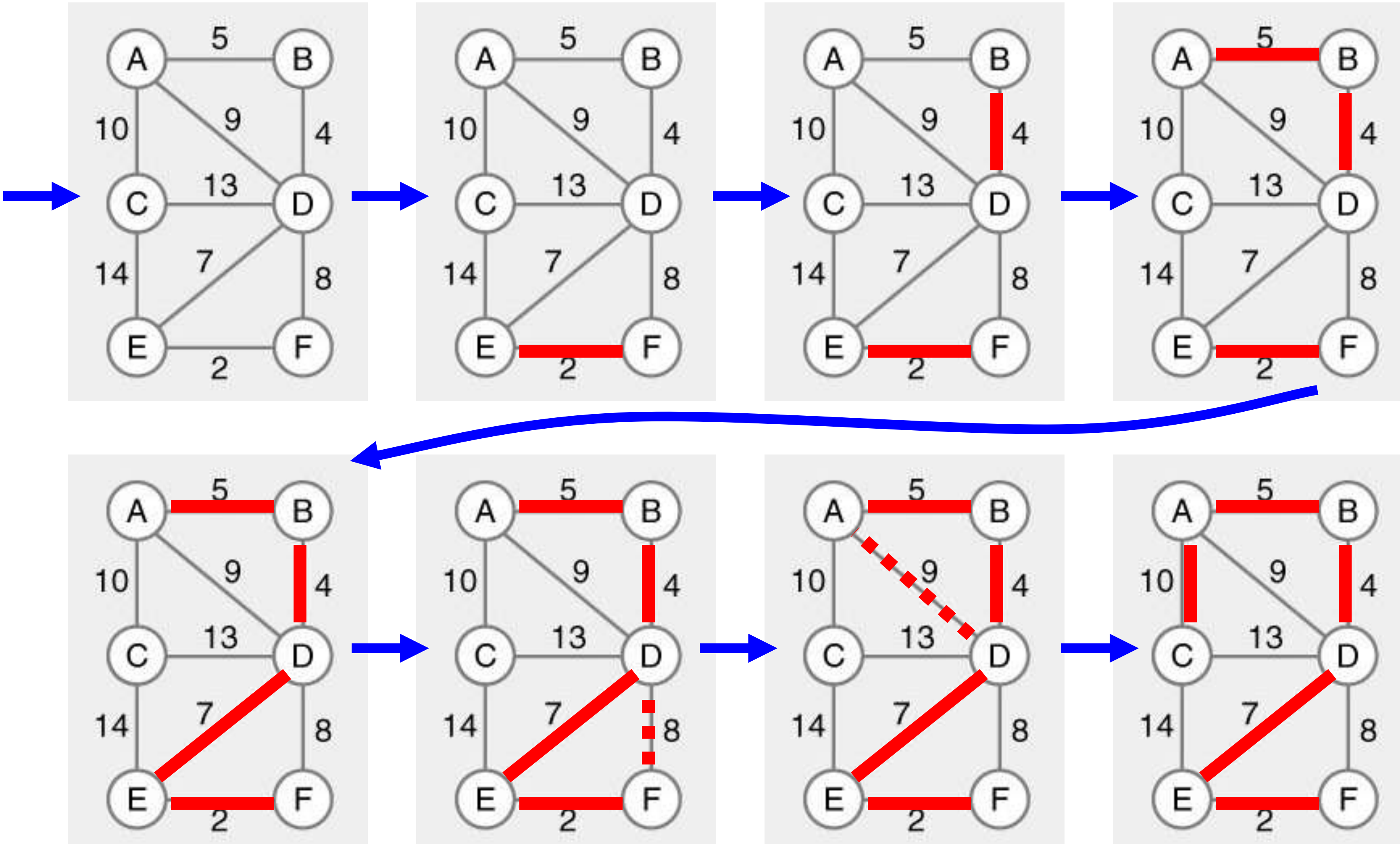
        stop when tree has N–1 edges

Keeps track of

- lightest edge remaining
- whether adding edge to MST creates cycle

Optimal solution computed with greedy algorithm

# MST – Kruskal's Algorithm Example



# MST – Kruskal's Algorithm

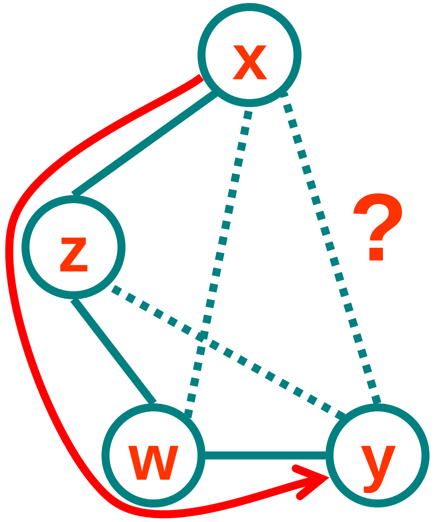
- When does adding (X,Y) to tree create cycle?

- Two approaches to finding cycles
  1. Traversal
  2. Connected subgraph

# MST – Kruskal's Algorithm

- Traversal approach
  - Traverse tree starting at X
  - If we can reach Y, adding (X,Y) would create cycle

- Example
  - Question
    - Add (X,Y) to MST?
  - Answer
    - No, since can already reach Y from X by traversing MST

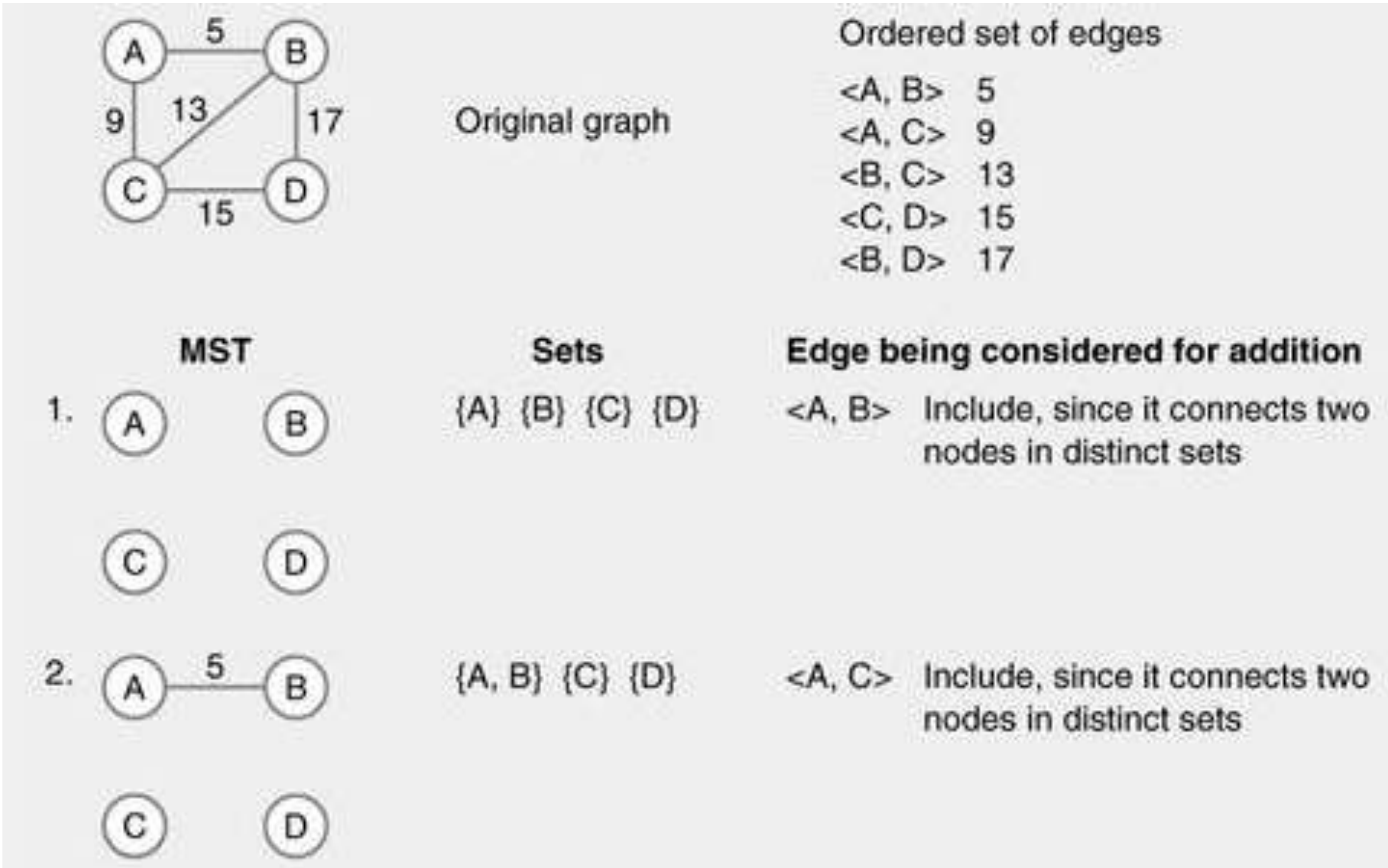


# MST – Kruskal’s Algorithm

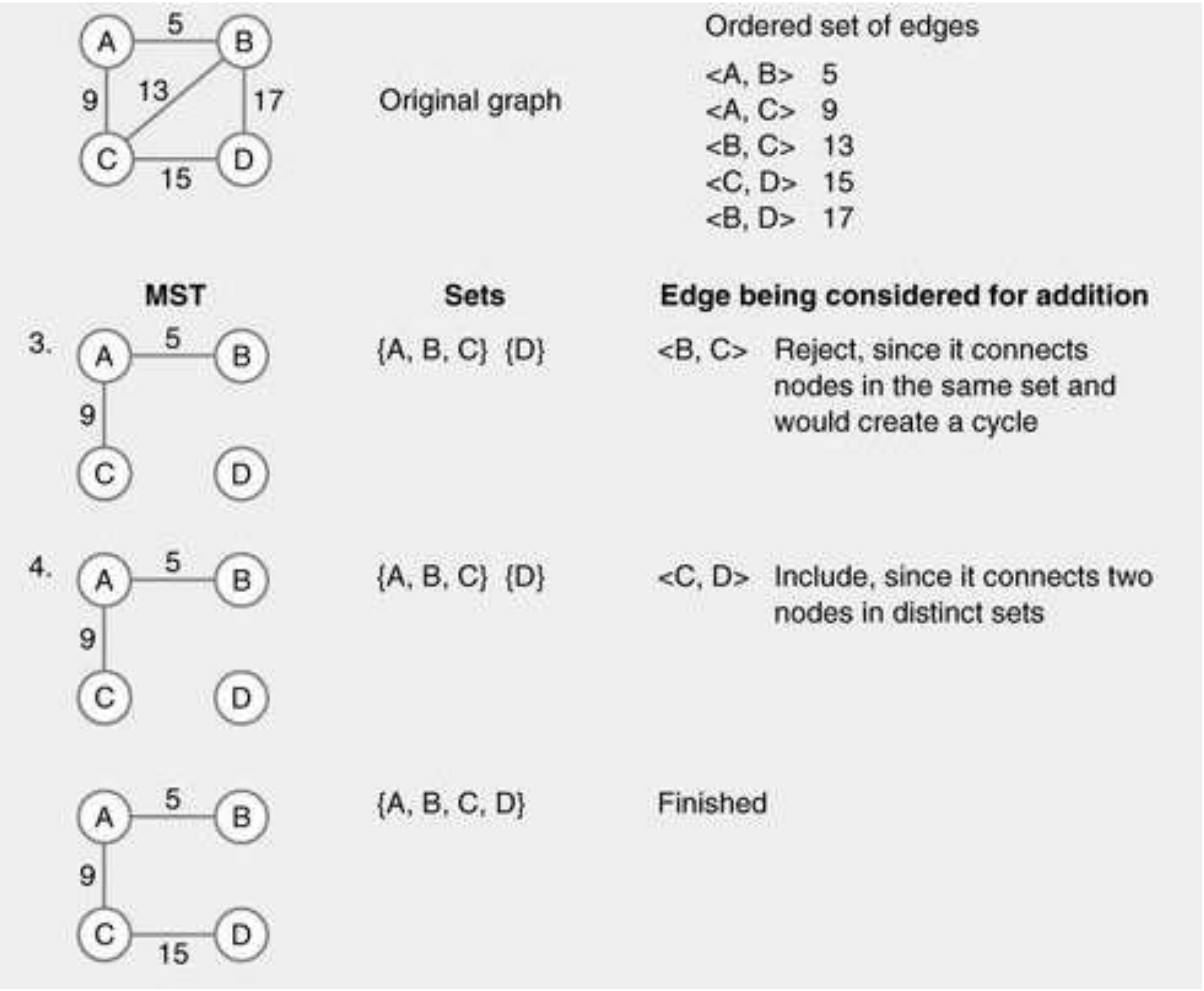
- Connected subgraph approach
  - Maintain set of nodes for each connected subgraph
  - Initialize one connected subgraph for each node
  - If X, Y in same set, adding (X,Y) would create cycle
  - Otherwise
    1. Add edge (X,Y) to spanning tree
    2. Merge sets containing X, Y

- To test set membership
  - Use Union-Find algorithm

# MST – Connected Subgraph Example



# MST – Connected Subgraph Example

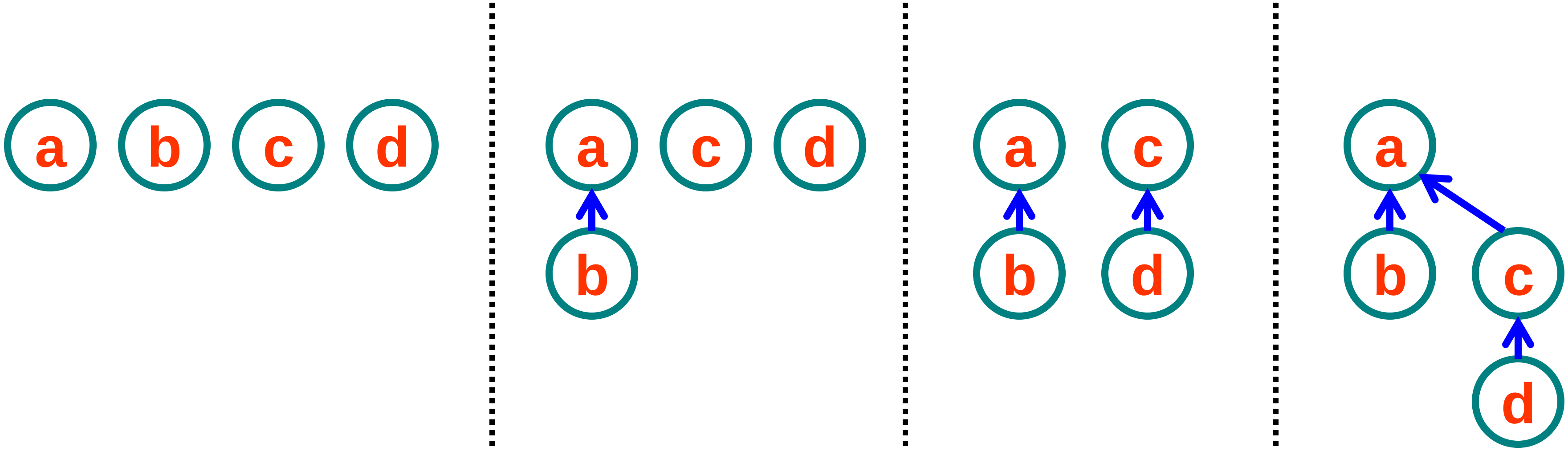


# Union-Find Algorithm

- Union-Find
  - Algorithm & data structure
  - Very efficient for testing membership in disjoint sets
- Problem description
  - Start with n nodes, each in different subgraph
  - Support two operations
    - Find – are nodes x & y in same subgraph?
    - Union – merge subgraphs containing x & y

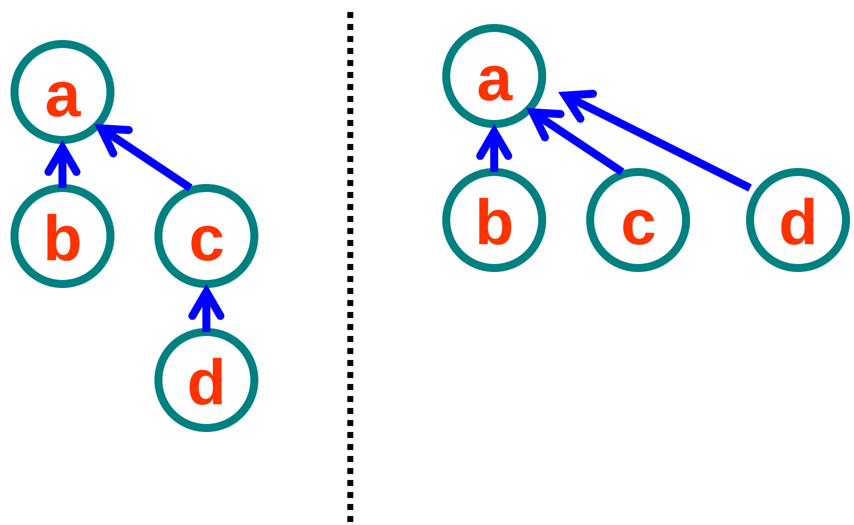
# Union-Find Algorithm

- **Basic approach**
  - Each node has a parent pointer
  - Find – follow parent pointer(s) to root of tree
  - Union – point root of 1$^{st}$ tree to root of 2$^{nd}$ tree
- **Example**
  - Union( a, b ) ; union( c , d); union( b, d)



# Union-Find Algorithm

- **Path compression**
  - Speeds up future Find( ) operations
    1. Follow parent pointer(s) to root of tree
    2. Update all nodes along path to point to root
- **Example**
  - Find(d)




So how fast is Union-Find?

# Ackermann's Function

- Function

```
int A(x,y) {
    if (x == 0)
      return y+1;
    if (y == 0)
      return A(x – 1, 1);
    return A(x – 1, A(x, y – 1));
}
```

- A( ) grows fast
  - $A(2,2) = 7$
  - $A(3,3) = 61$
  - $A(4,2) = 2^{65536} - 3$
  - $A(4,3) = 2^{2^{65536}} - 3$
  - $A(4,4) = 2^{2^{2^{65536}}} - 3$

# Inverse Ackermann's Function

- Definition
  - $\alpha(n)$ is the inverse Ackermann's function
  - $\alpha(n)$ = the smallest k such that $A(k,k) \geq n$
- Fun fact
  - $\alpha$(number of atoms in universe) = 4
- Union-find
  - A sequence of n operations requires $O(n\ \alpha(n))$ time
  - Practically speaking, indistinguishable from $O(n)$

# Graph Summary

- Graph data structure
  - Very useful in practice
  - Different representations

- Many graph algorithms
  - Traversal
  - Shortest path
  - Minimum spanning tree