CMSC 132: OBJECT-ORIENTED PROGRAMMING II

Algorithmic Complexity I

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Algorithm Efficiency

- Efficiency
  - Amount of resources used by algorithm
    - Time, space
- Measuring efficiency
  - Benchmarking
    - Approach
      - Pick some desired inputs
      - Actually run implementation of algorithm
      - Measure time & space needed
  - Asymptotic analysis

Benchmarking

- Advantages
  - Precise information for given configuration
    - Implementation, hardware, inputs
- Disadvantages
  - Affected by configuration
    - Data sets (often too small)
      - Dataset that was the right size 3 years ago is likely too small now
    - Hardware
    - Software
  - Affected by special cases (biased inputs)
  - Does not measure intrinsic efficiency

Asymptotic Analysis

- Approach
  - Mathematically analyze efficiency
  - Calculate time as function of input size n
    - $T \approx O(f(n))$
    - T is on the order of $f(n)$
    - “Big O” notation
- Advantages
  - Measures intrinsic efficiency
  - **Dominates efficiency for large input sizes**
  - Programming language, compiler, processor irrelevant

Search Example

- Number guessing game
  - Pick a number between 1…n
  - Guess a number
  - Answer “correct”, “too high”, “too low”
  - Repeat guesses until correct number guessed

Linear Search Algorithm

- Algorithm
  - Guess number = 1
  - If incorrect, increment guess by 1
  - Repeat until correct
- Example
  - Given number between 1…100
  - Pick 20
  - Guess sequence = 1, 2, 3, 4 … 20
  - Required 20 guesses

Linear Search Algorithm

- Analysis of # of guesses needed for 1…n
  - If number = 1, requires 1 guess
  - If number = n, requires n guesses
  - On average, needs n/2 guesses
  - Time = O(n) = Linear time

Binary Search Algorithm

- Algorithm
  - Set low and high to be lowest and highest possible value
  - Guess middle = (low+high)/2
  - If too large, set high = middle-1
  - If too small, set low = middle+1
  - Repeat until guess correct

Binary Search Algorithm

- Example
  - Given number between 1…100
  - Secret number we are trying is find is 20
  - Guesses
    - low = 1, high = 100, guess 50, Answer = too large
    - low = 1, high = 49, guess 25, Answer = too large
    - low = 1, high = 24, guess 12, Answer = too small
    - low = 13, high = 24, guess 18, Answer = too small
    - low = 19, high = 24, guess 21, Answer = too large
    - low = 19, high = 20, guess 19, Answer = too small
    - low = 20, high = 20, guess 20, Answer = correct
    - Required 7 guesses

Binary Search Algorithm

- Analysis of # of guesses needed for 1…n
  - If number = n/2, requires 1 guess
  - If number = 1, requires $\log_2(n)$ guesses
  - If number = n, requires $\log_2(n)$ guesses
  - On average, needs $\log_2(n)$ guesses
  - Time = $O(\log_2(n)) = O(\log(n))$ = Log time

Search Comparison

- For number between 1…100
  - Simple algorithm = 50 steps
  - Binary search algorithm = $\log_2(n)$ = 7 steps
- For number between 1…100,000
  - Simple algorithm = 50,000 steps
  - Binary search algorithm = $\log_2(n)$ (about 17 steps)
- Binary search is much more efficient!

Asymptotic Complexity

- Comparing two linear functions

Size	Running Time	
	n/2	4n+3
64	32	259
128	64	515
256	128	1027
512	256	2051

Asymptotic Complexity

- Comparing two functions
  - n/2 and 4n+3 behave similarly
  - Run time roughly doubles as input size doubles
  - Run time increases linearly with input size
- For large values of n
  - Time(2n) / Time(n) approaches exactly 2
- Both are O(n) programs

Asymptotic Complexity

- Comparing two log functions

Size	Running Time	
	$\log_2(n)$	$5 * \log_2(n) + 3$
64	6	33
128	7	38
256	8	43
512	9	48

Asymptotic Complexity

- Comparing two functions
  - $\log_2(n)$ and $5 * \log_2(n) + 3$ behave similarly
  - Run time roughly increases by constant as input size doubles
  - Run time increases logarithmically with input size
- For large values of n
  - Time(2n) – Time(n) approaches constant
  - Base of logarithm does not matter
    - Simply a multiplicative factor

      $\log_a N = (\log_b N) / (\log_b a)$
- Both are O(log(n)) programs

Asymptotic Complexity

- Comparing two quadratic functions

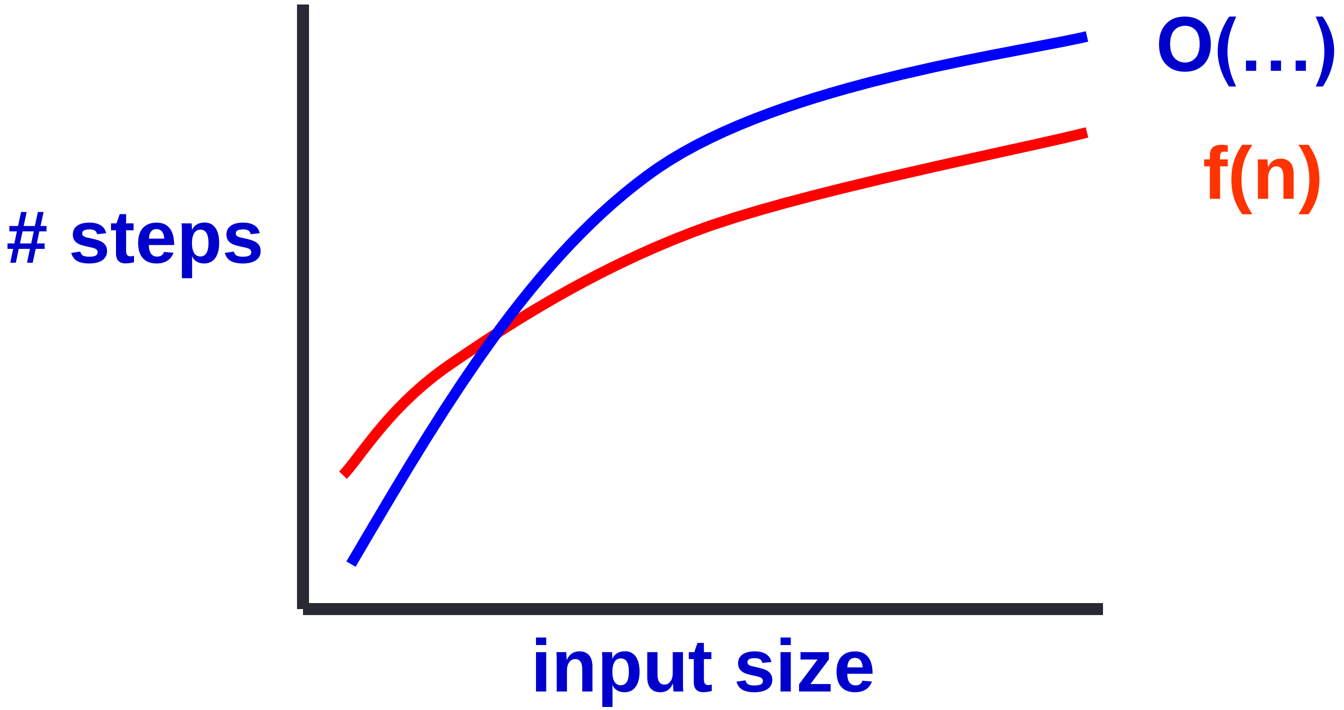
Size	Running Time	
	n^2	$2\ n^2 + 8$
2	4	16
4	16	40
8	64	132
16	256	520

Asymptotic Complexity

- Comparing two functions
  - n^2 and $2\ n^2 + 8$ behave similarly
  - Run time roughly increases by 4 as input size doubles
  - Run time increases quadratically with input size
- For large values of n
  - Time(2n) / Time(n) approaches 4
- Both are O(n^2) programs

Big-O Notation

- Represents
  - Upper bound on number of steps in algorithm
    - For sufficiently large input size
  - Intrinsic efficiency of algorithm for large inputs



Formal Definition of Big-O

- Function f(n) is O(g(n)) if
  - For some positive constants M, N_0
  - $M \times g(n) \geq f(n)$, for all $n \geq N_0$
- Intuitively
  - For some coefficient M & all data sizes $\geq N_0$
    - $M \times g(n)$ is always greater than f(n)

Big-O Examples

- $5n + 1000 \Rightarrow O(n)$
  - Select $M = 6$, $N_0 = 1000$
  - For $n \geq 1000$
    - $6n \geq 5n+1000$ is always true
  - Example $\Rightarrow$ for $n = 1000$
    - $6000 \geq 5000 + 1000$

Big-O Examples

- $2n^2 + 10n + 1000 \Rightarrow O(n^2)$
  - Select $M = 4$, $N_0 = 100$
  - For $n \geq 100$
    - $4n^2 \geq 2n^2 + 10n + 1000$ is always true
  - Example $\Rightarrow$ for $n = 100$
    - $40000 \geq 20000 + 1000 + 1000$

Observations

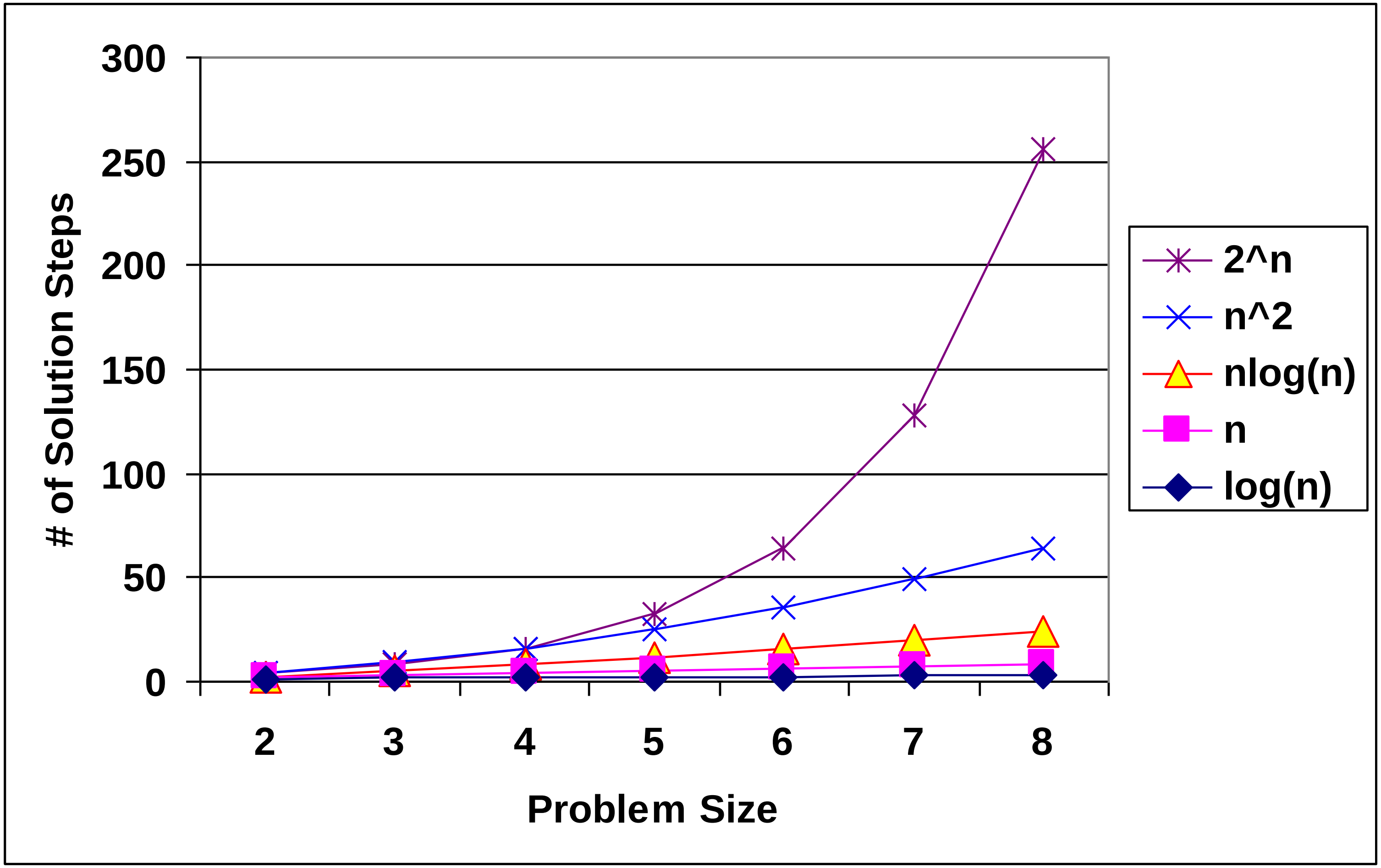
- For large values of n
  - Any O(log(n)) algorithm is faster than O(n)
  - Any O(n) algorithm is faster than O(n^2)
- Asymptotic complexity is fundamental measure of efficiency

Asymptotic Complexity Categories

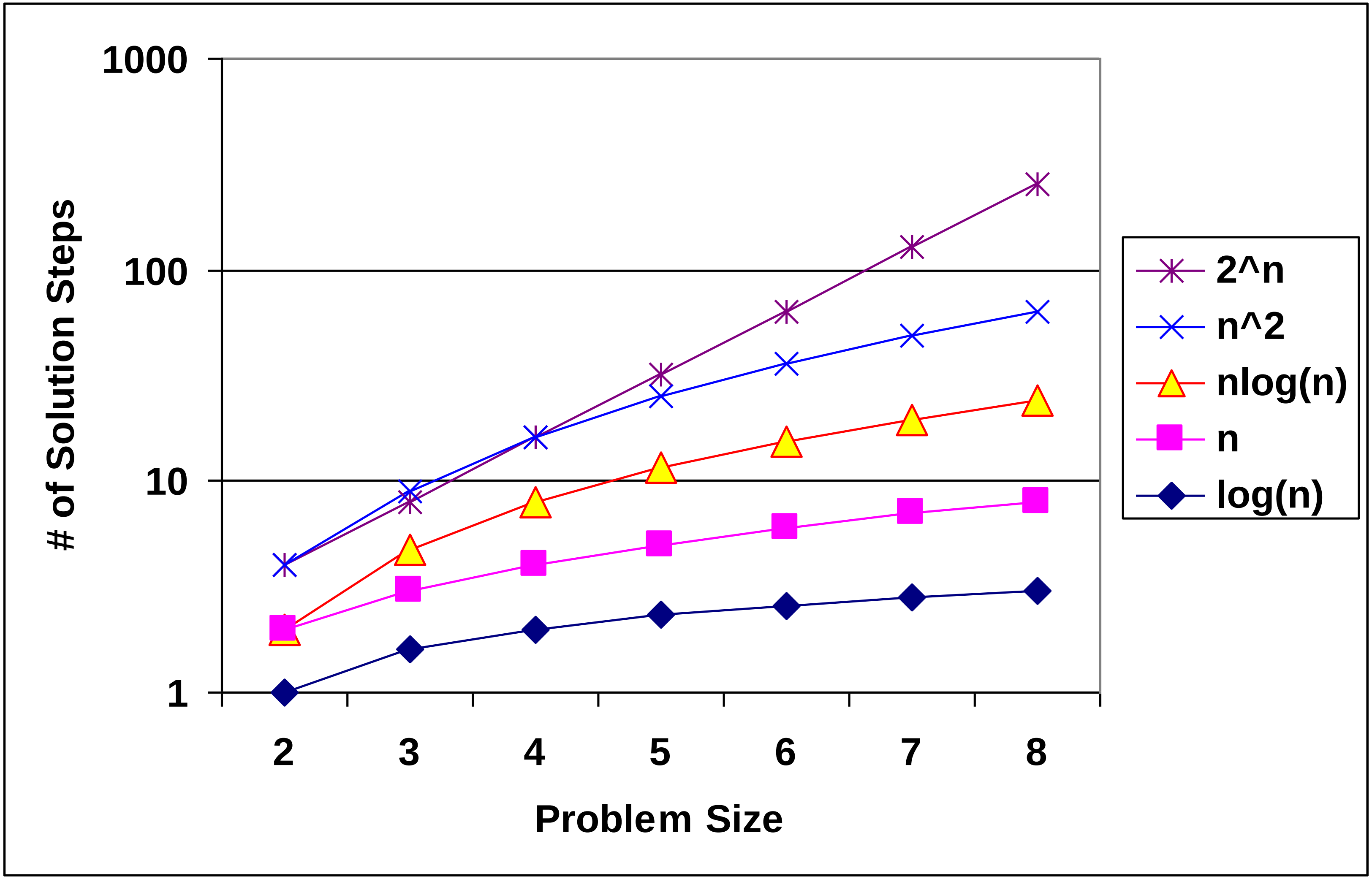
Complexity	Name	Example
• $O(1)$	Constant	Array access
• $O(\log(n))$	Logarithmic	Binary search
• $O(n)$	Linear	Largest element
• $O(n \log(n))$	N log N	Optimal sort
• $O(n^2)$	Quadratic	2D Matrix addition
• $O(n^3)$	Cubic	2D Matrix multiply
• $O(n^k)$	Polynomial	Linear programming
• $O(k^n)$	Exponential	Integer programming
• $O(n!)$	Factorial	Brute-force search TSP
• $O(n^n)$	N to the N	

From smallest to largest, for size n, constant $k > 1$

Complexity Category Example



Complexity Category Example

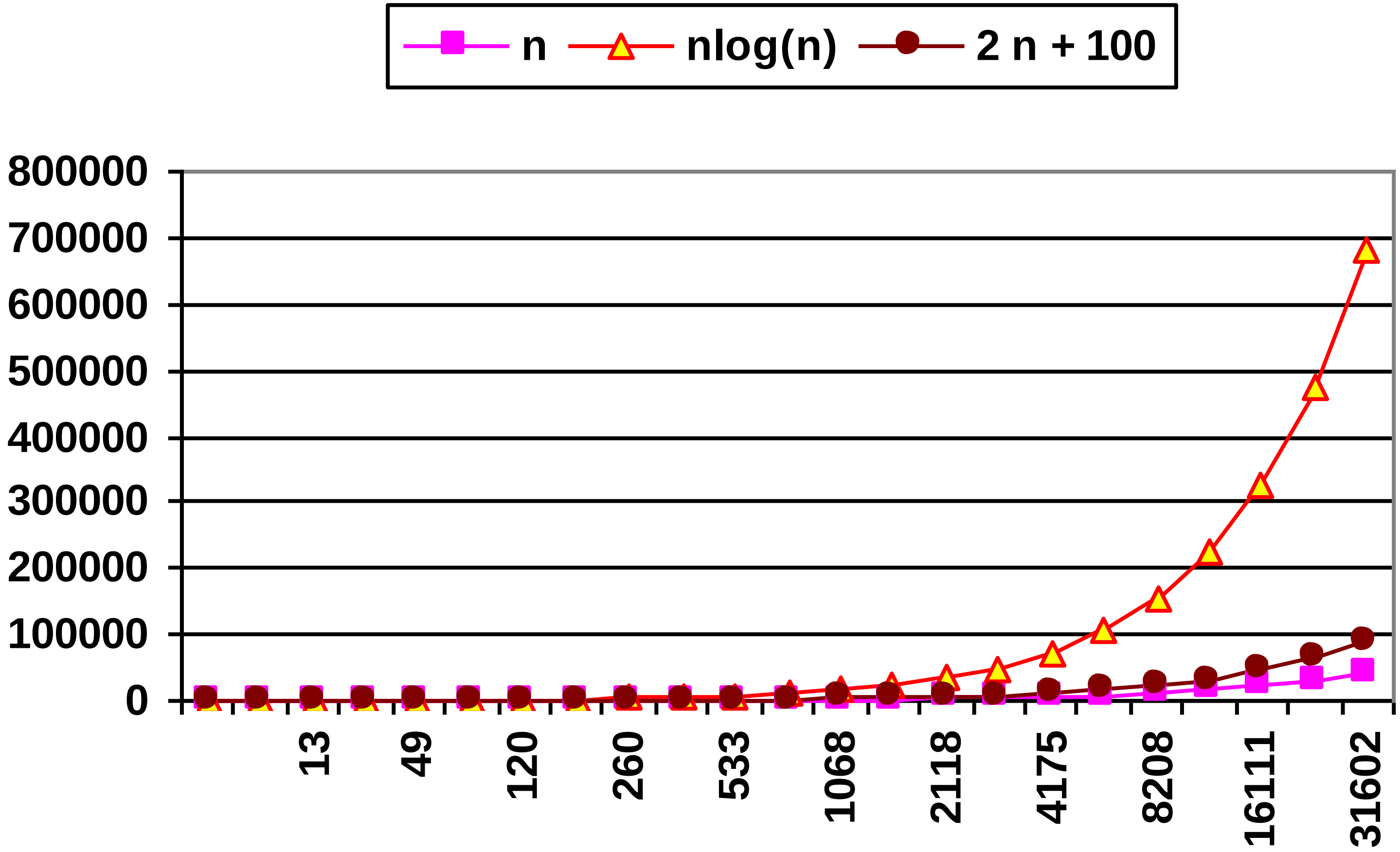


Calculating Asymptotic Complexity

- As **n** increases
  - Highest complexity term dominates
  - Can ignore lower complexity terms
- Examples
  - $2n + 100 \Rightarrow O(n)$
  - $10n + n\log(n) \Rightarrow O(n\log(n))$
  - $100n + \frac{1}{2}n^2 \Rightarrow O(n^2)$
  - $100n^2 + n^3 \Rightarrow O(n^3)$
  - $1/100\,2^n + 100n^4 \Rightarrow O(2^n)$

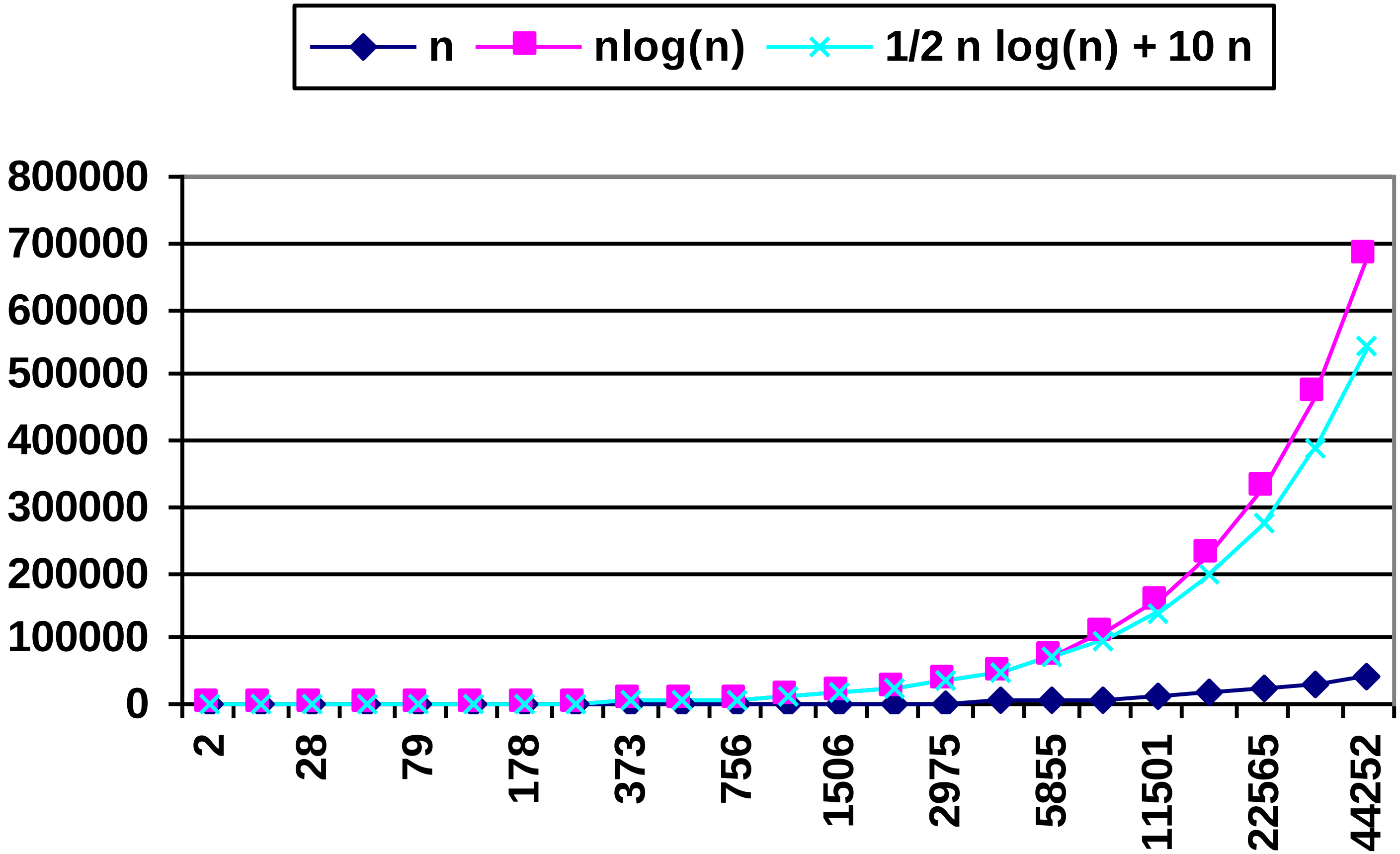
Complexity Examples

- $2n + 100 \quad \Rightarrow O(n)$



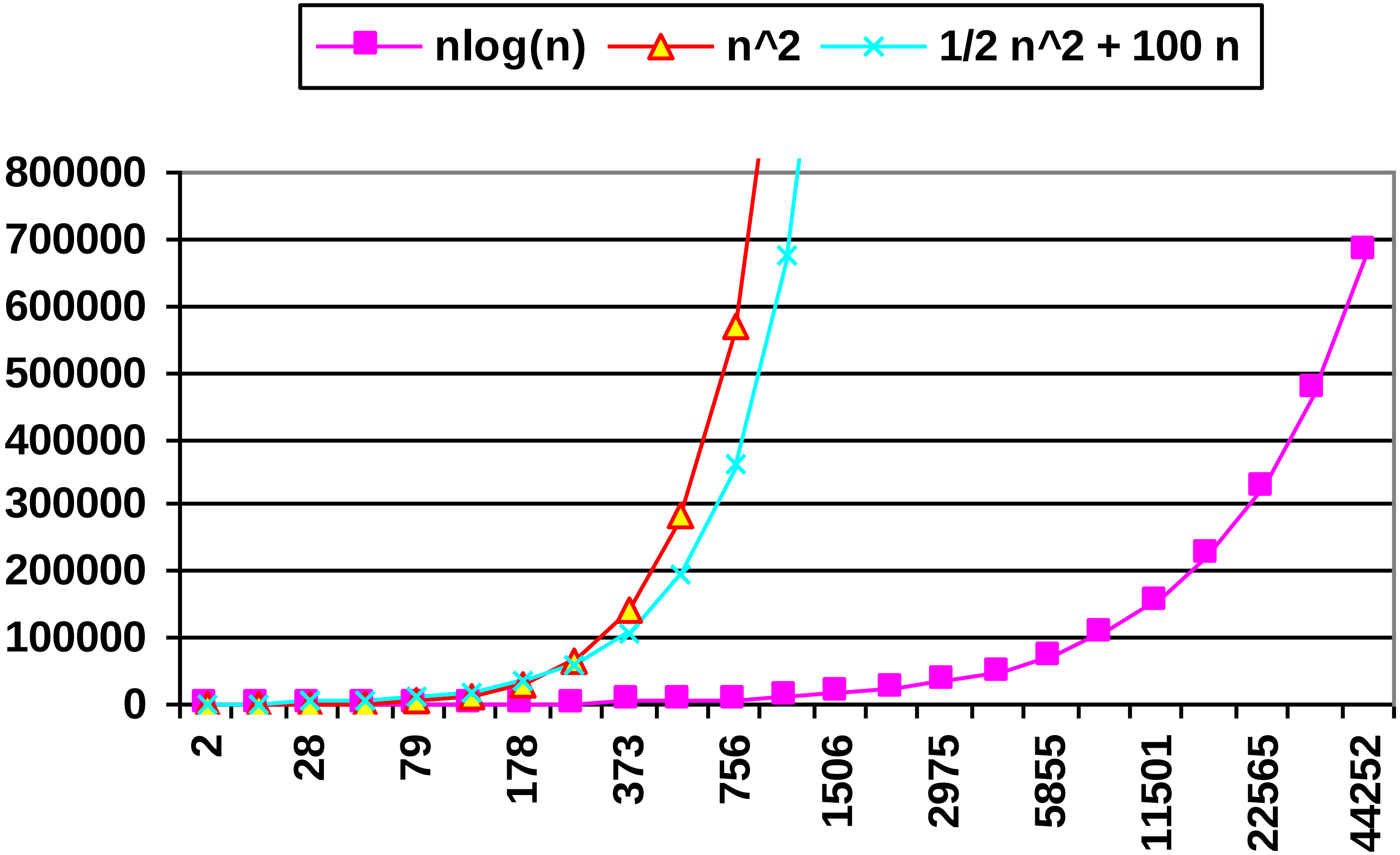
Complexity Examples

- ½ n log(n) + 10 n $\Rightarrow$ O(nlog(n))



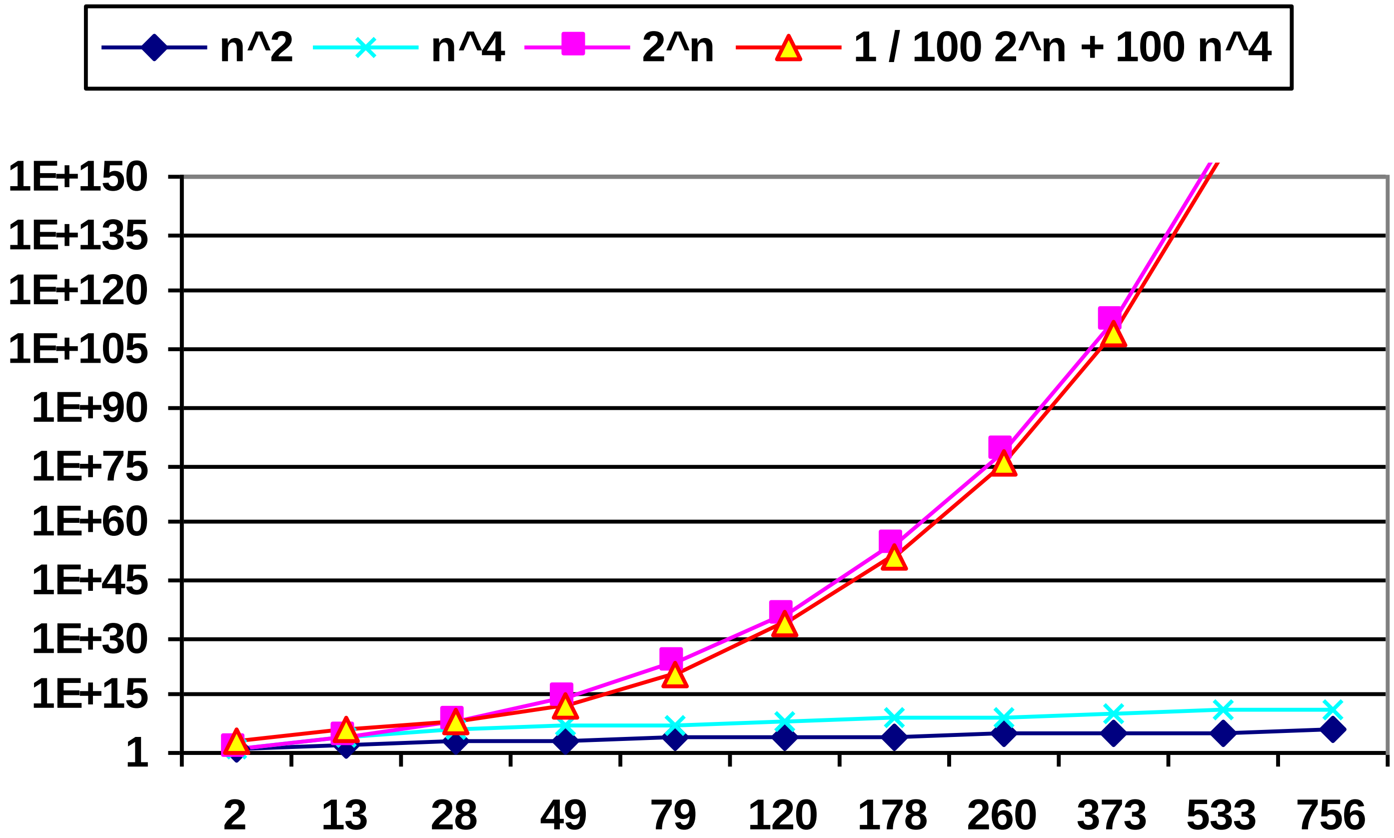
Complexity Examples

- $\frac{1}{2} n^2 + 100 n \Rightarrow O(n^2)$



Complexity Examples

- $1/100\ 2^n + 100\ n^4 \Rightarrow O(2^n)$



Types of Case Analysis

- Can analyze different types (cases) of algorithm behavior
- Types of analysis
  - Best case
  - Worst case
  - Average case
  - Amortized

Types of Case Analysis (Best/Worst)

- Best case
  - Smallest number of steps required
  - Not very useful
  - Example $\Rightarrow$ Find item in first place checked
- Worst case
  - Largest number of steps required
  - Useful for upper bound on worst performance
    - Real-time applications (e.g., multimedia)
    - Quality of service guarantee
  - Example $\Rightarrow$ Find item in last place checked

Quicksort Example

- Quicksort
  - One of the fastest comparison sorts
  - Frequently used in practice
- Quicksort algorithm
  - Pick pivot value from list
  - Partition list into values smaller & bigger than pivot
  - Recursively sort both lists
- Quicksort properties
  - Average case = $O(n\log(n))$
  - Worst case = $O(n^2)$
    - Pivot ≈ smallest / largest value in list
    - Picking from front of nearly sorted list
- Can avoid worst-case behavior
  - Select random pivot value

Types of Case Analysis (Average)

- Average case analysis
  - Number of steps required for “typical” case
  - Most useful metric in practice
  - Different approaches: average case, expected case
- Average case
  - Average over all possible inputs
    - Assumes all inputs have the same probability
  - Example
    - Case 1 = 10 steps, Case 2 = 20 steps
    - Average = 15 steps
- Expected case
  - Weighted average over all possible inputs
    - Based on probability of each input
  - Example
    - Case 1 (90%) = 10 steps, Case 2 (10%) = 20 steps
    - Average = 11 steps

Amortized Analysis

- Approach
  - Applies to worst-case sequences of operations
  - Finds average running time per operation
  - Example
    - Normal case = 10 steps
    - Every 10^{th} case may require 20 steps
    - Amortized time = 11 steps
- Assumptions
  - Can predict possible sequence of operations
  - Know when worst-case operations are needed
    - Does not require knowledge of probability
- By using amortized analysis we can show the best way to grow an array is by doubling its size (rather than increasing by adding one entry at a time)