CMSC 132: OBJECT-ORIENTED PROGRAMMING II

Heaps & Priority Queues

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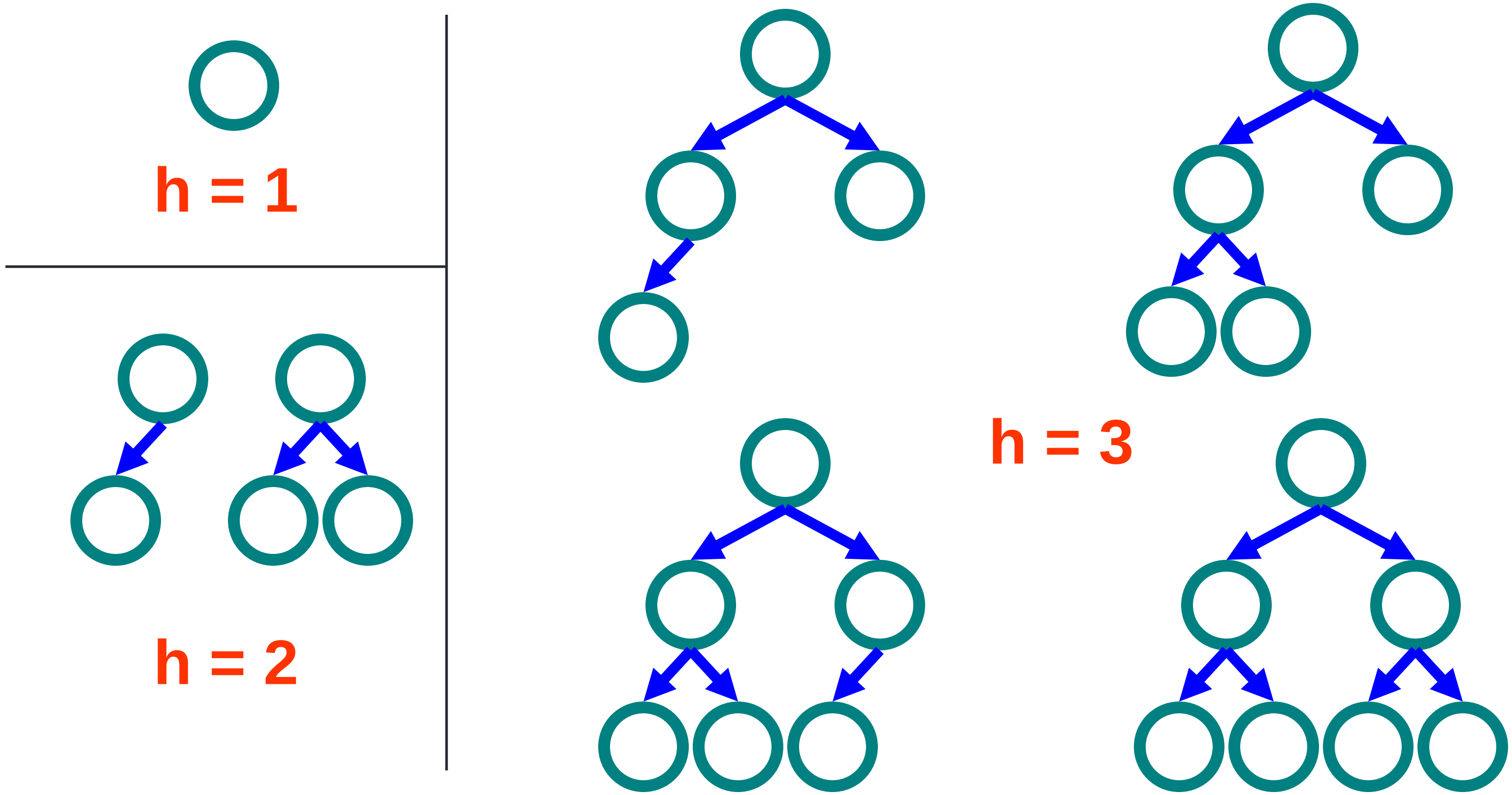
Overview

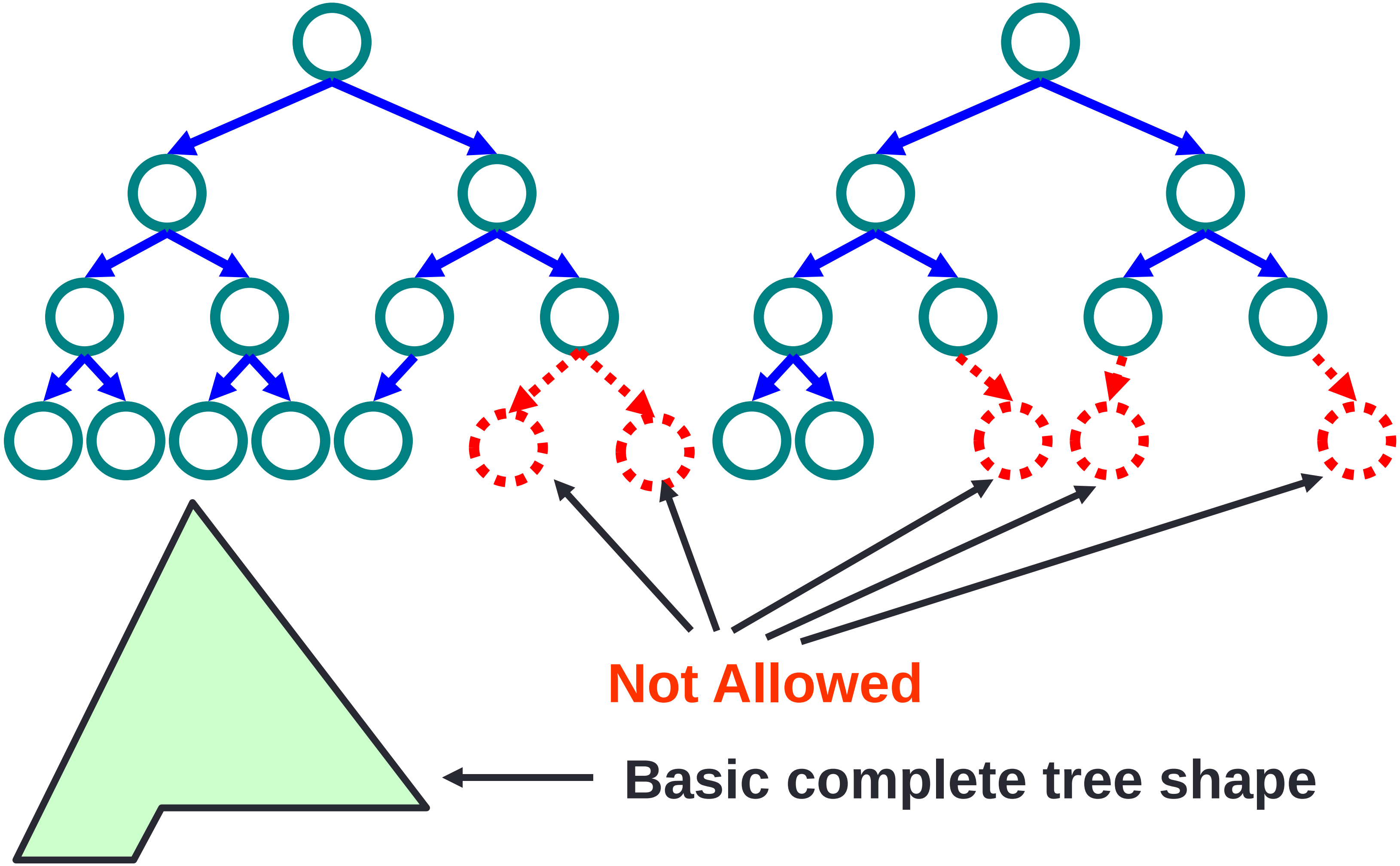
- Binary trees
  - Complete
- Heaps
  - Insert
  - getSmallest
- Heap applications
  - Heapsort
  - Priority queues

Complete Binary Trees

- An binary tree (height h) where
  - Perfect tree to level h-1
  - Leaves at level h are as far left as possible

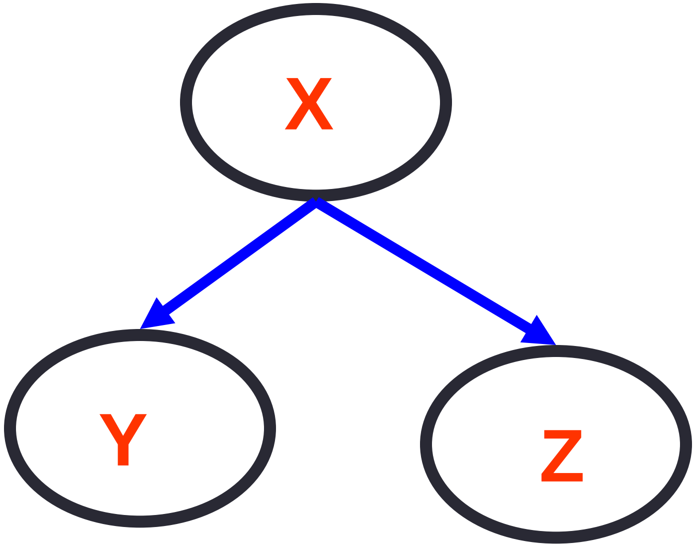


Complete Binary Trees

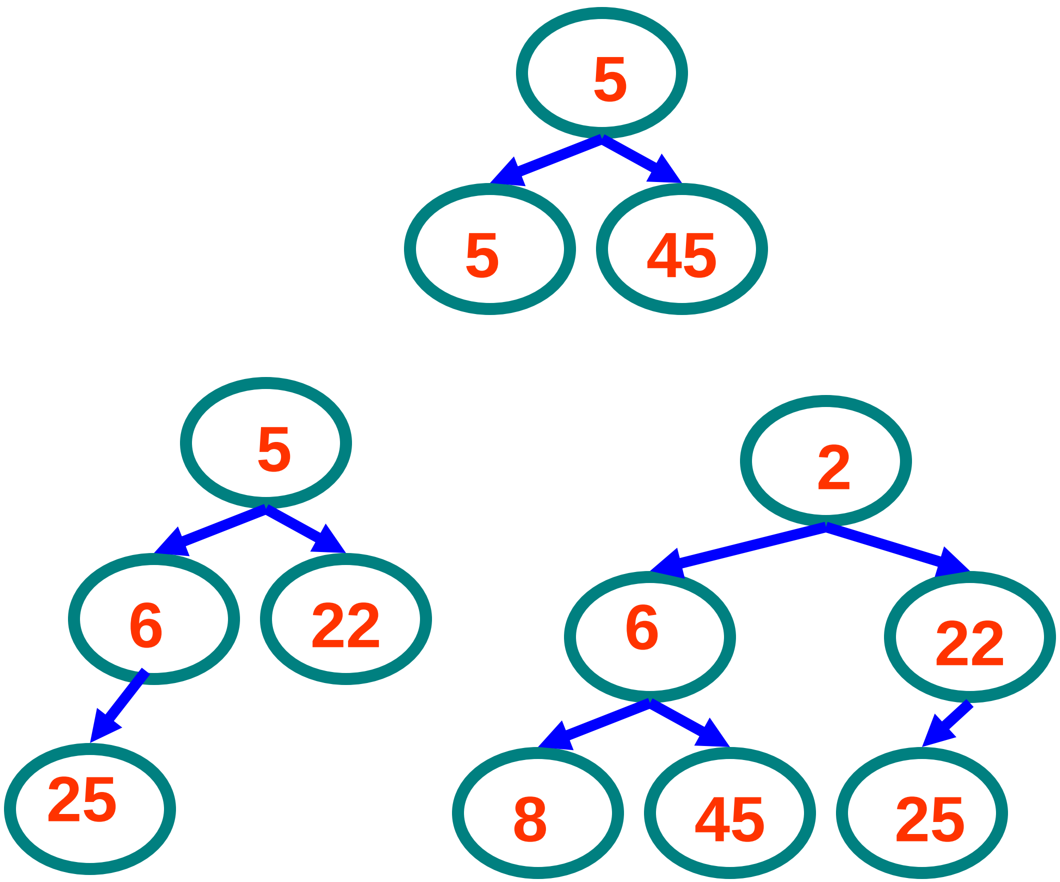


Heaps

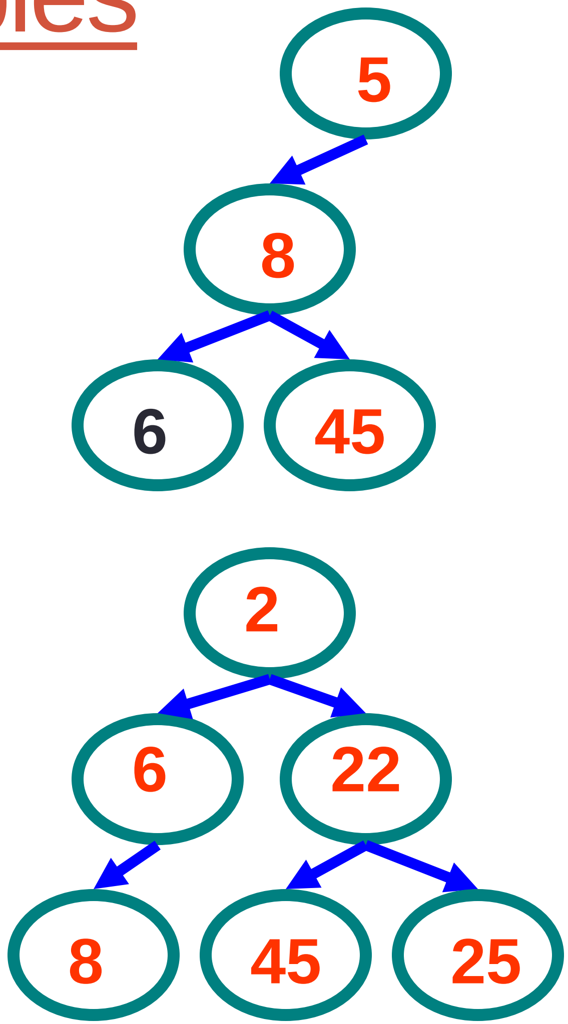
- Two key properties
  - Complete binary tree
  - Value at node
    - Smaller than or equal to values in subtrees
- Example heap
  - $X \leq Y$
  - $X \leq Z$



Heap & Non-heap Examples



Heaps



Non-heaps

Heap Properties

- Heaps are balanced trees
  - Height = $\log_2(n) = O(\log(n))$

- Can find smallest element easily
  - Always at top of heap!

- Can organize heap to find maximum value
  - Value at node larger than values in subtrees
  - Heap can track either min or max, but not both

Heap

- Key operations
  - Insert (X)
  - getSmallest ( )

- Key applications
  - Heapsort
  - Priority queue

Heap Operations – Insert(X)

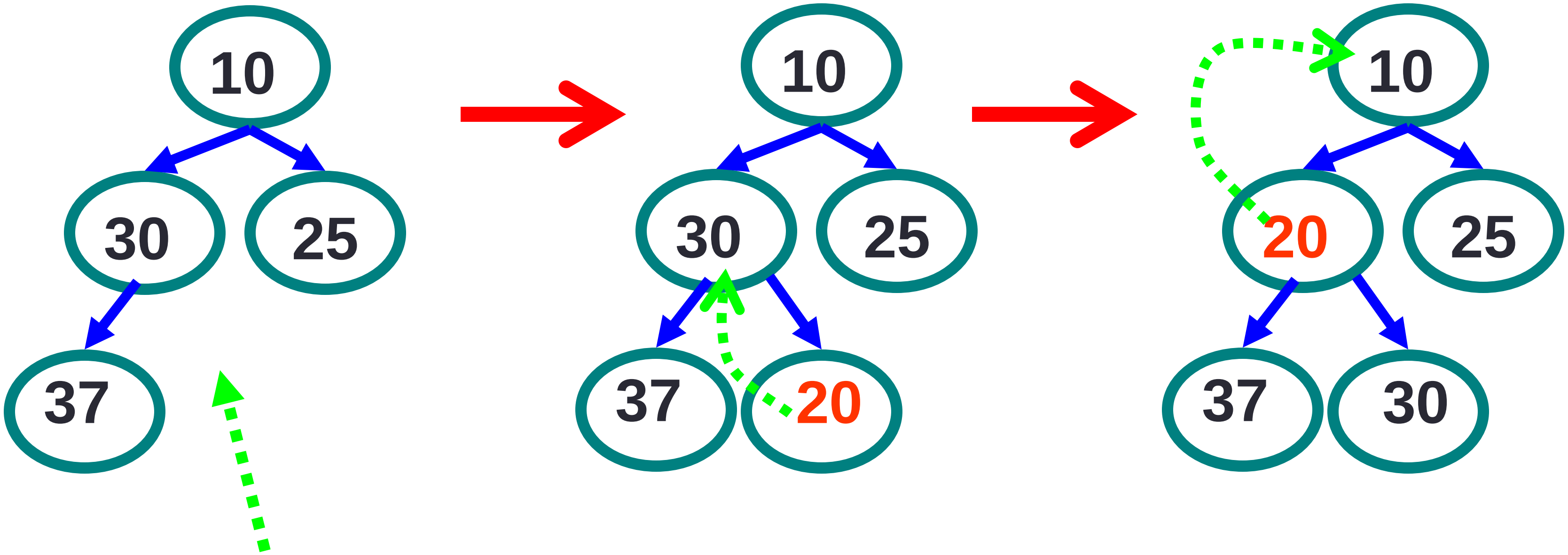
- Algorithm
  - Add X to end of tree
  - While (X < parent)

    Swap X with parent // X bubbles up tree

- Complexity
  - # of swaps proportional to height of tree
  - O(log(n))

Heap Insert Example

- Insert (20)



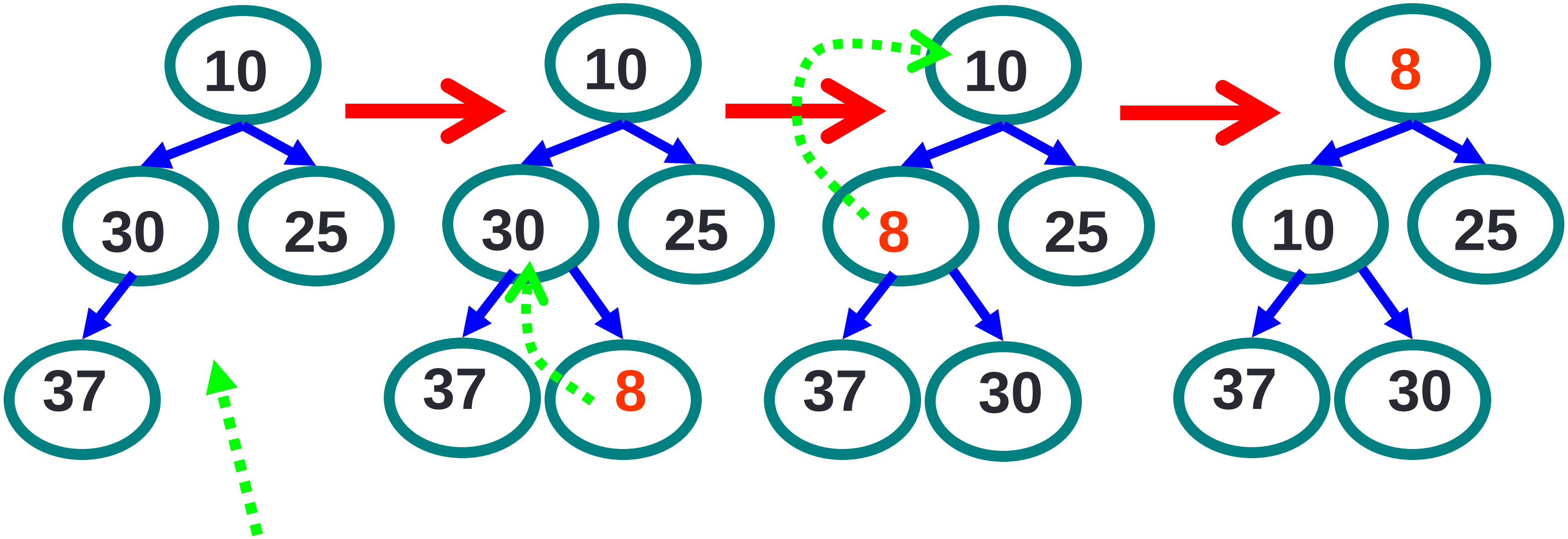
1) Insert to end of tree

2) Compare to parent, swap if parent key larger

3) Insert complete

Heap Insert Example

- Insert (8)



1) Insert to end of tree

2) Compare to parent, swap if parent key larger

3) Insert complete

Heap Operation – getSmallest()

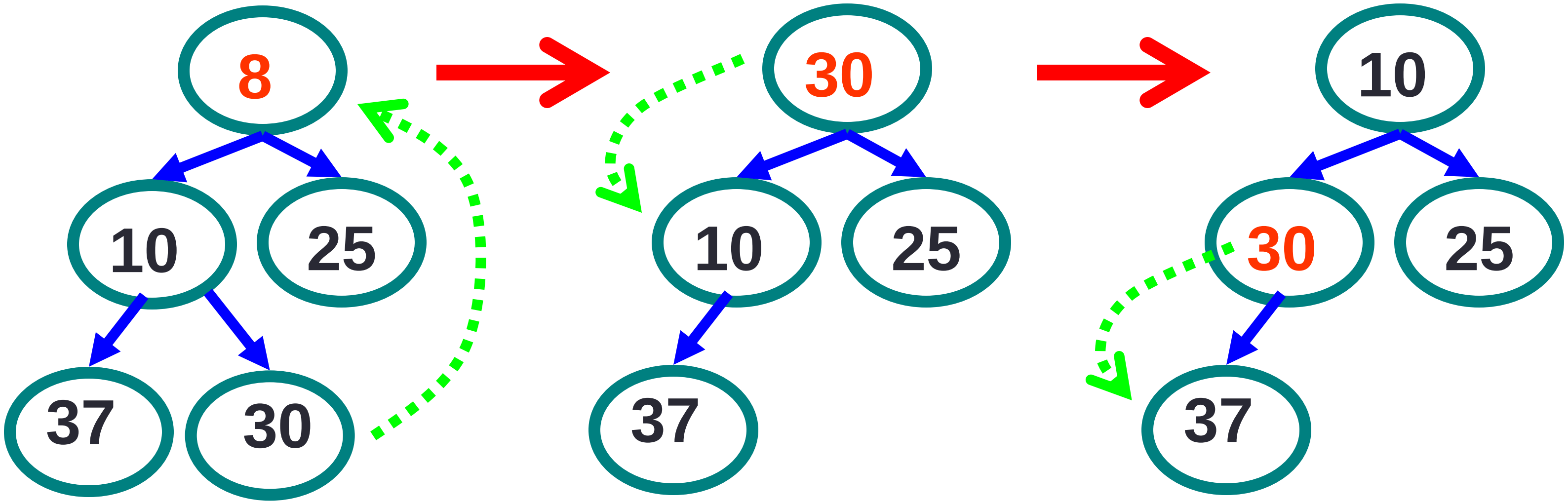
- Algorithm
  - Get smallest node at root
  - Replace root with X at end of tree
  - While (X > child)

    Swap X with smallest child // X drops down tree
  - Return smallest node

- Complexity
  - # swaps proportional to height of tree
  - O(log(n))

Heap GetSmallest Example

- getSmallest ()



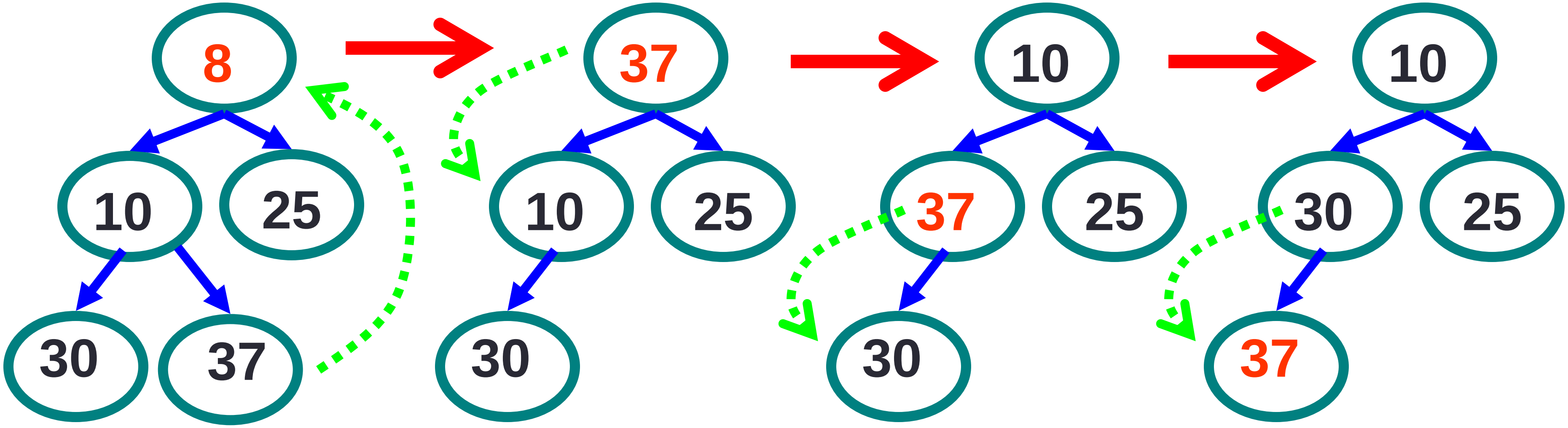
1) Replace root with end of tree

2) Compare node to children, if larger swap with smallest child

3) Repeat swap if needed

Heap GetSmallest Example

- getSmallest ()



1) Replace root with end of tree

2) Compare node to children, if larger swap with smallest child

3) Repeat swap if needed

Heap Implementation

- Can implement heap as array
  - Store nodes in array elements
  - Assign location (index) for elements using formula

(a) Heap represented as a tree

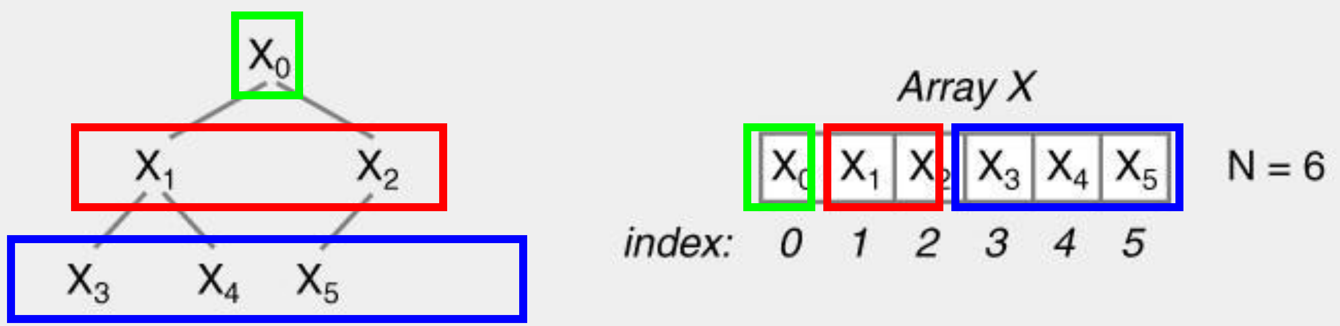
(b) Heap represented as an array

Heap Implementation

- Observations
  - Compact representation
  - Edges are implicit (no storage required)
  - Works well for complete trees (no wasted space)

Heap Implementation

- Calculating node locations
  - Array index i starts at 0
  - Parent(i) = $\lfloor (i - 1) / 2 \rfloor$
  - LeftChild(i) = $2 \times i + 1$
  - RightChild(i) = $2 \times i + 2$

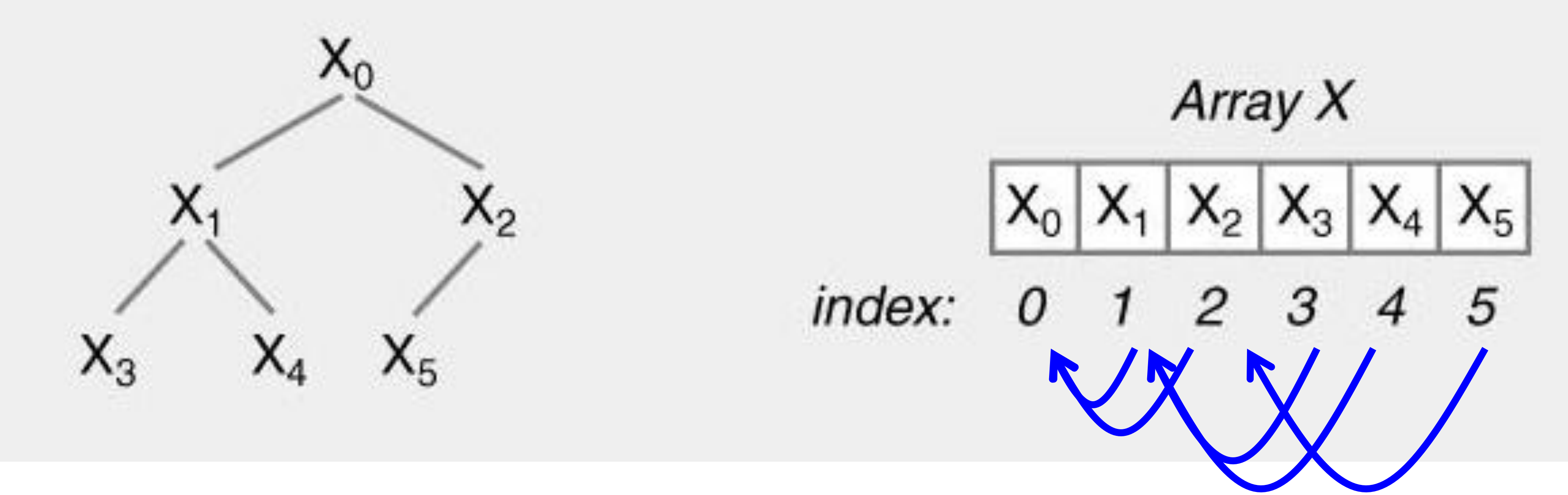


(a) Heap represented as a tree

(b) Heap represented as an array

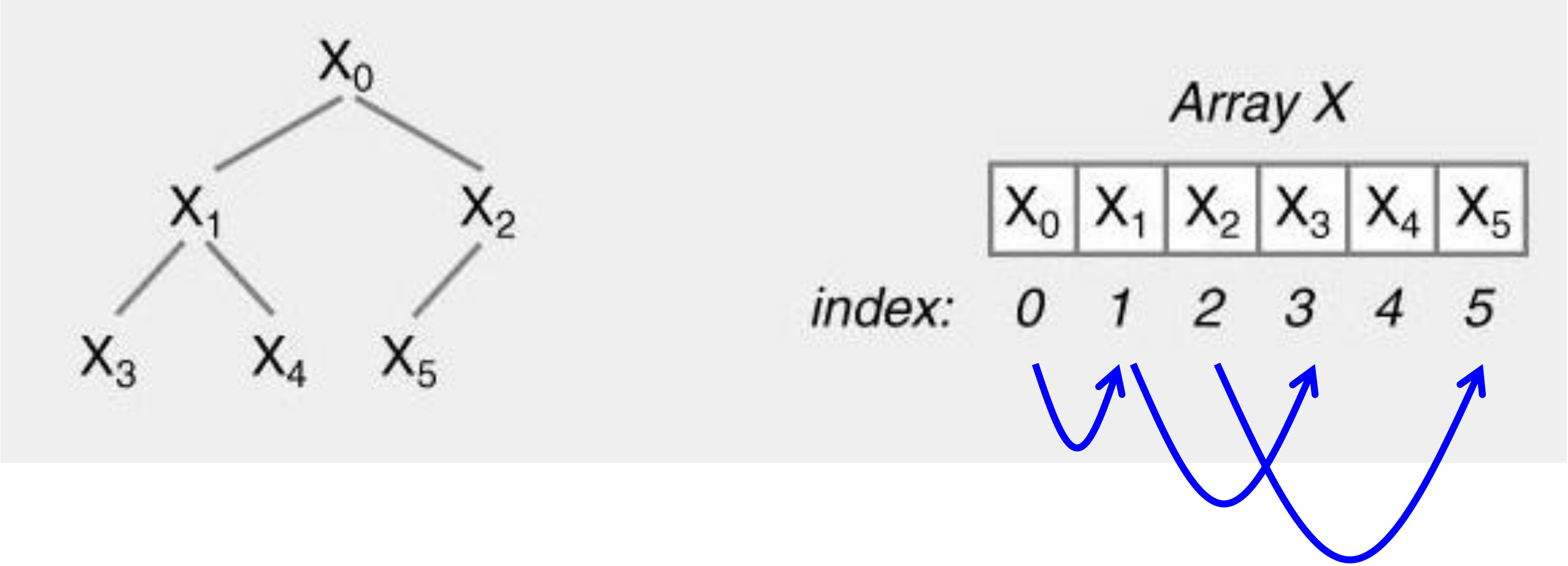
Heap Implementation

- Example
  - Parent(1) = ⌊ (1 – 1) / 2 ⌋ = ⌊ 0 / 2 ⌋ = 0
  - Parent(2) = ⌊ (2 – 1) / 2 ⌋ = ⌊ 1 / 2 ⌋ = 0
  - Parent(3) = ⌊ (3 – 1) / 2 ⌋ = ⌊ 2 / 2 ⌋ = 1
  - Parent(4) = ⌊ (4 – 1) / 2 ⌋ = ⌊ 3 / 2 ⌋ = 1
  - Parent(5) = ⌊ (5 – 1) / 2 ⌋ = ⌊ 4 / 2 ⌋ = 2



Heap Implementation

- Example
  - LeftChild(0) = $2 \times 0 + 1 = 1$
  - LeftChild(1) = $2 \times 1 + 1 = 3$
  - LeftChild(2) = $2 \times 2 + 1 = 5$



Heap Implementation

- Example
  - RightChild(0) = 2 × 0 +2 = 2
  - RightChild(1) = 2 × 1 +2 = 4

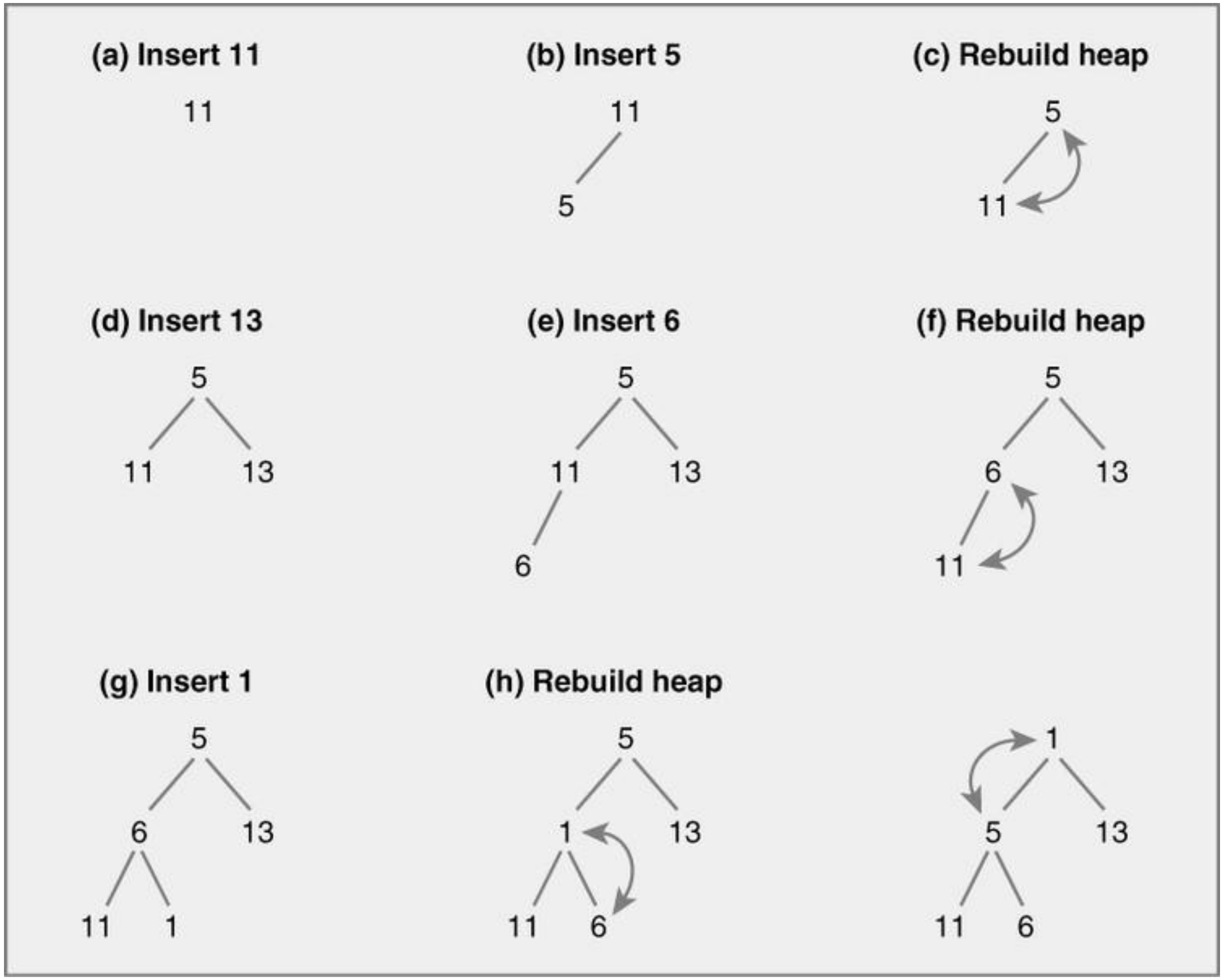
Heap Application – Heapsort

- Use heaps to sort values
  - Heap keeps track of smallest element in heap
- Algorithm
  1. Create heap
  2. Insert values in heap
  3. Remove values from heap (in ascending order)
- Complexity
  - O(nlog(n))

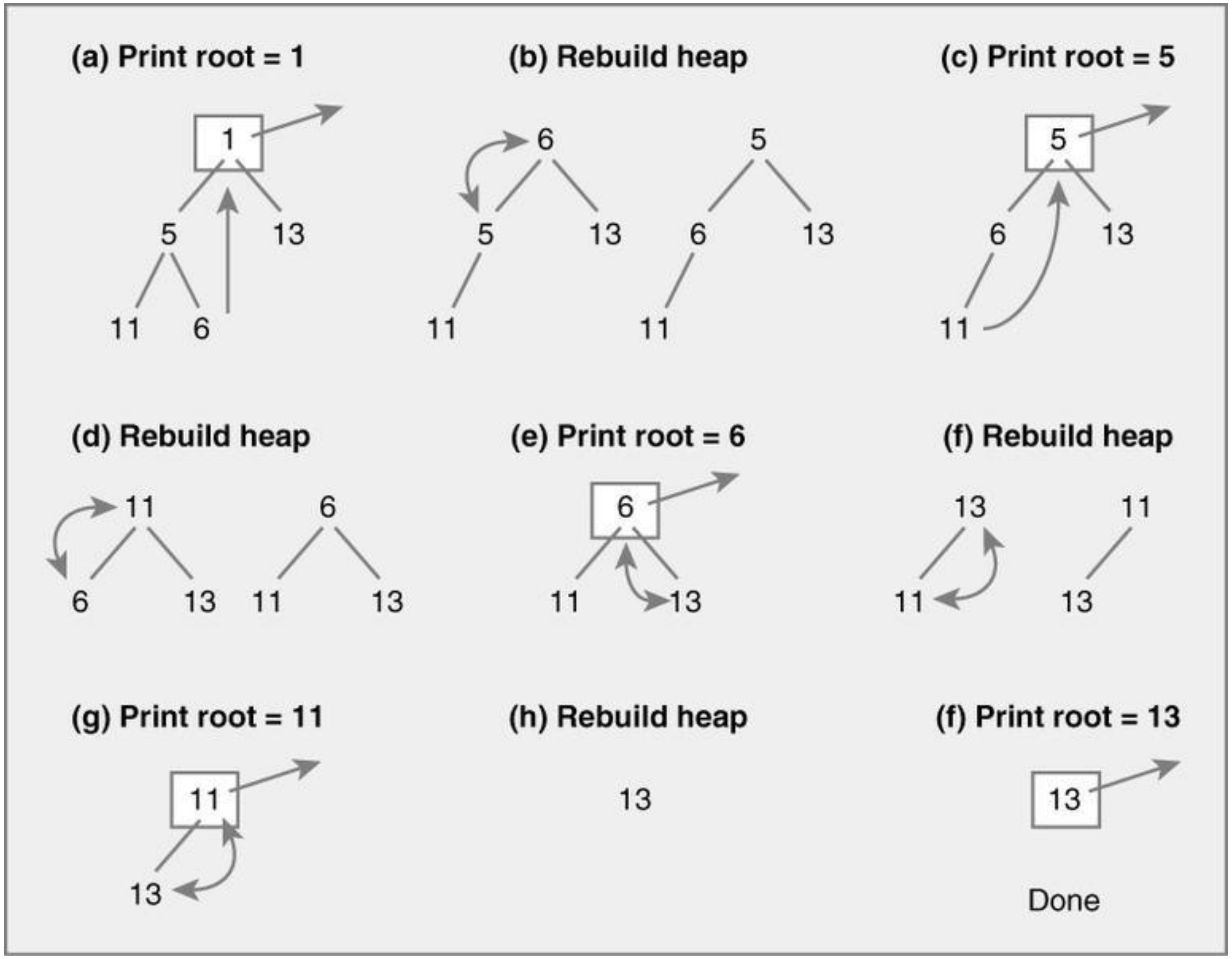
Heapsort Example

- Input
  - 11, 5, 13, 6, 1
- View heap during insert, removal
  - As tree
  - As array

Heapsort – Insert Values



Heapsort – Remove Values



Heapsort – Insert in to Array 1

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Insert 11	11				

Heapsort – Insert in to Array 2

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Insert 5	11	5			
Swap	5	11			

Heapsort – Insert in to Array 3

- Input
    - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Insert 13	5	11	13		

Heapsort – Insert in to Array 4

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Insert 6	5	11	13	6	
Swap	5	6	13	11	

…

Heapsort – Remove from Array 1

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Remove root	1	5	13	11	6
Replace	6	5	13	11	
Swap w/ child	5	6	13	11	

Heapsort – Remove from Array 2

- Input
  - 11, 5, 13, 6, 1

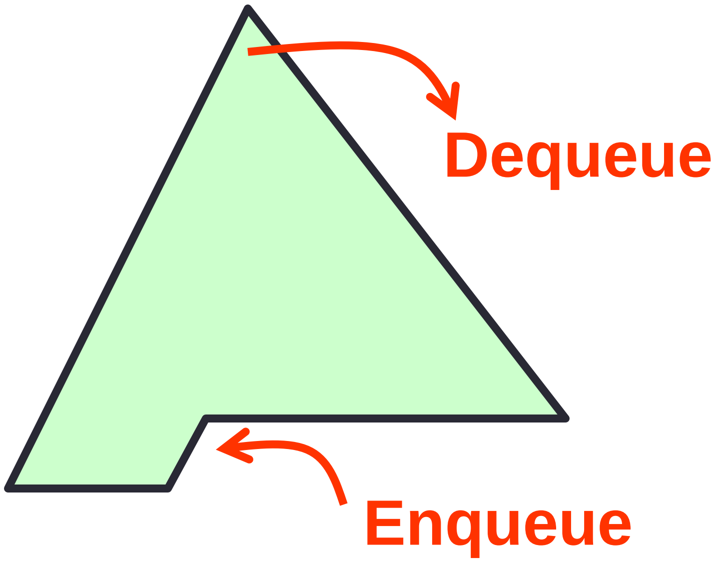
Index =	0	1	2	3	4
Remove root	5	6	13	11	
Replace	11	6	13		
Swap w/ child	6	11	13		

Heap Application – Priority Queue

- Queue
  - Linear data structure
  - First-in First-out (FIFO)
  - Implement as array / linked list

Heap Application – Priority Queue

- Priority queue
  - Elements are assigned priority value
  - Higher priority elements are taken out first
  - Implement as heap
    - Enqueue $\Rightarrow$ insert( )
    - Dequeue $\Rightarrow$ getSmallest( )



Priority Queue

- Properties
  - Lower value = higher priority
  - Heap keeps highest priority items in front
- Complexity
  - Enqueue $\Rightarrow$ insert( ) = O(log(n))
  - Dequeue $\Rightarrow$ getSmallest( ) = O(log(n))
  - For any heap

Heap vs. Binary Search Tree

- Binary search tree
  - Keeps values in sorted order
  - Find any value
    - O(log(n)) for balanced tree
    - O(n) for degenerate tree (worst case)
- Heap
  - Keeps smaller values in front
  - Find minimum value
    - O(log(n)) for any heap