EQUIVALENCE RELATIONS

Recall that "a equiv_n b" means n | b-a, i.e., there exists an integer k such that b-a = nk.

One can think of this in terms of "sticks". Consider a stick of length n (cms, say). If we can lay out that stick over and over, starting at a and ending exactly at b, then a and b are a multiple of n apart. All the integers that can be reached this way, starting from integer a, are those that are "equivalent to a mod n".

This concept -- as we saw much earlier -- partitions Z into the subsets
nZ
1+nZ
2+nZ
...
(n-1) + nZ

When n is 3, we get
3Z: {... -6 -3 0 3 6 ...} -- multiples of 3, aka integers equiv to 0 mod 3
1+3Z: {... -5 -2 1 4 7 ...} -- one more than multiples of 3, aka integers equiv to 1 mod 3
2+3Z: {... -4 -1 2 5 8 ...} -- two more than multiples of 3, aka integers equiv to 2 mod 3

This partition of Z is { 3Z, 1+3Z, 2+3Z }. Every integer is in one -- and only one -- of the three partition elements. The integers within any *one* of those subsets are considered "equivalent" to one another: they are all reachable from one another by sticks of length n; put differently, they all give the same remainder when divided by n. Elements of a single partition subset are "the same" in some respect.

Such a situation can vastly be generalized to *any* partition of *any* set.

Definition: Let S be a set, and P = (S1, S2, ... } a partition of S. We can consider the elements of any one partition subset as *equivalent* to each other, in this sense: they all belong to the same partition subset!

Imagine S as a large set of people, and P as some breaking up of S into "teams"; this can be done in any way at all, even at random. But once the teams are formed, members of the same team are the same in that sense: they belong to the same team.

Here is another illustration: Let S = {0,1,2,...,9,10} and let P consist of the subsets
{0,9}
{2,4,6,7}
{1,8}
{3}
{5,10}
Then P partitions S: the subsets are disjoint and their union is S. And this means that -- as far as this partition goes -- 2, 4,6, and 7 are all equivalent to each other; also 5 and 10; 1 and 8; and 0 and 9. And 3 is equivalent only to itself. True, there is no arithmetical connection here, making 1 and 8 "the same" in some way; the sameness is by accident of which subset they belong to. It's as if, say, 1 and 8 live in France; 2, 4, 6, and 7 in Japan; etc; with no other similarity.

What good is it, then? Lots! There are a great many situations in which a set naturally partitions in some way or other (by arithmetic, by happenstance, by history, by temperature, by design, etc}. Moreover, there are underlying "rules" that govern elements of the same partition subset, no matter what the partition is based on. Three such rules are these:

1. symmetry: if a is equivalent to b, then be is equivalent to a [that is, if a belongs to the same partition subset as b, then b belongs to the same partition subset as a]

2. reflexivity: a is equivalent to a [that is, a belongs to the same partition subset as a]

3. transitivity: if a is equivalent to b, and b to c, then a is equivalent to c [that is, if a belongs to the same subset as b, and b the same as c, then a belongs to the same subset as c]

These are trivially obvious, so much so that is seems a waste of tine to even mention them. But they are important, and lead to this key notion:

Definition: Given a set S, an "equivalence relation" on S is a set E of ordered pairs <x,y> of elements x,y of S having the "SRT" properties of

a. symmetry: for all x,y in S, if <x,y> in E, then <y,x> in E [aka: if x equiv y, then y equiv x]

b. reflexivity: <x,x> in E for all x in S [aka: x equiv x]

c. transitivity: for all x,y,x in S, if <x,y> in E and <y,z> in E, then <x,z> in E [aka: x equiv y and y equiv z implies x equiv z]

The set of ALL pairs of elements of S is just the Cartesian Product of S with itself: SxS = { <x,y> : x,y in S }. So an equiv relation E as above is a special subset of SxS, namely one satisfying the "SRT" conditions.

The above discussion -- carried a bit further -- leads to the following general result:

Theorem: Given a set S, for every partition P of S there a unique equivalence relation E on S (namely E = {<x,y> : x and y belong to the same partition subset}), and conversely: for every equiv relation on S, there is a partition P such that each partition subset (known as an equivalence class) is a subset of S containing all elts of S equivalent to some fixed elt of S. [Thus for example, all integers equiv mod 5 to, say, 3 -- this gives the subset 3+5Z].

Thus there can be a great many equivalence relations on a set, namely as many as there can be partitions of the set.

Example: Let S be the set {a,b,c}. There are 5 partitions of S:
P1: {a,b,c}
P2: {a,b} {c}
P3: {a,c} {b}
P4: {a} {b,c}
P5: {a} {b} {c}
And each corresponds to a distinct equivalence relation:
E1 = SxS
E2 = {<a,b>, <b,a>, <a,a>, <b,b>, <c,c>}
E3 = {<a,c>, <c,a>, <a,a>, <b,b>, <c,c>}
E4 = {<b,c>, <c,b>, <a,a>, <b,b>, <c,c>}
E5 = {<a,a>, <b,b>, <c,c>}

Which of these are equivalence relations; and for those that are, what are the partitions?
1. x<=y, on Z
2. x<y, on Z
3. x and y have exactly the same molecular weight, on a set of chemical samples
4. x and y are within 0.001 gm of each other's weights, on the set of rocks
5. x and y have the same floor, on the set of reals
6. x and y live in the same country, on the set of people
7. x and y live within 100 miles of each other, on the set of people
8. x=y, on any set S.
9. x^2 = y^2, on the set R of reals.
10. d/dx f = d/dx g, on the set of differentiable functions from R to R
11. d/dx f >= d/dx g, on the same set as in 10.
12. mean-runtime(p) = mean-runtime(q), on the set of JAVA programs

13. either p and q both eventually halt, or p and q both have infinite loops, on the same set as in 12
14. either p and q have at most 100 lines of code, or both have more than 100, same set as in 12

RELATIONS IN GENERAL

Definition: A (binary) relation on a set S is any subset T of SxS.

So, an equivalence relation on S is a relation on S, namely one that satisfies the SRT conditions.

But there are more relations (on a given set S) than equivalence relations, simply because there are always some relations that are not equivalence relations. The simplest example is the empty relation: emptyset is a subset of SxS; and it is not an equiv-rel since it violates reflexivity (as long as S is non-empty: any b in S would require <b,b> to be in the equiv-rel).

Example: Let S = {b,c}. Then these are all 16 possible relations on S:
1. emptyset
2. {<b,b>}
3. {<b,c>}
4. {<c,c>}
5. {<c,b>}
6. {<b,b>,<b,c>}
7. {<b,b>,<c,c>}
8. {<b,b>,<c,b>}
9. {<b,c>,<c,c>}
10. {<b,c>,<c,b>}
11. {<c,c>,<c,b>}
12. {<b,b>,<b,c>,<c,c>}
13. {<b,b>,<b,c>,<c,b>}
14. {<b,b>,<c,c>,<c,b>}
15. {<b,c>,<c,c>,<c,b>}
16. {<b,b>,<b,c>,<c,c>,<c,b>}

Which of them are equiv-relations, and what are their associated partitions?
Ans:
7; {b},{c}
16: {b,c}

(Binary) relations that are not equiv-rels are very common, and include standard "everyday" ones such as:
<=, on a set of numbers
<, on numbers
=/=, on numbers
parent-of, on people
x^2 + y^2 <= 2, on the set of reals x and y

Note that a relation need not follow a meaningful pattern; it can consist of any pairings whatsoever, such as: {<3,2>,<9,9>,<pi,-72>,<0,-4>}. What is the "relationship" in such a case? Simply that the numbers have been paired up that way! This is the same principle that we saw for partitions: we can group together elts of a set into whatever "families" (disjoint subsets) we want, and then elts in the same subset are "related" by being in the same subset (that is, because we "said so").

Notation: If R is a relation on S (i.e., R subst of SxS) we write
<x,y> in R

or
xRy
to mean the same thing: that x is related to y by the relation R.

Note that we also sometimes speak of a binary relation from set A to set B; this is a subset of AxB.
Examples:

owner-of, from the set of people to the set of pets: { <person,pet>: person owns pet }

integral_from_0_to_1 of f, from the set reals to the set of continuous functions:
{ < Integral_0_to_1 f(x)dx , f > : f is a continuous function }

When A=B, then we have the case we defined initially.