# Counting and Probability

# Counting

- Counting elements in a list:
  - how many integers in the list from 1 to 10?
  - how many integers in the list from $m$ to $n$? (assuming $m \leq n$)

# How Many in a List?

- How many positive three-digit integers are there?
  - (this means only the ones that require 3 digits)
  - 999 – 99 = 900   (999 3 or fewer digit numbers – 99 2 or fewer)
  - 999 – 100 + 1 = 900   (100, 101, …, 999 – previous slide)
  - $9 \bullet 10 \bullet 10 = 900$   (9 hundreds digits, 10 tens digits, 10 unit digits)

- How many three-digit integers are divisible by 5?
  - $20 \bullet 5$, $21 \bullet 5$, …, $199 \bullet 5$
  - count the integers between 20 and 199
  - 199 – 20 + 1 = 180

# The breakfast problem

- Bill eats Rice Krispies, Cornflakes, Raisin Bran, or Cheerios.
- Bill drinks coffee, orange juice, or milk.
- How different types of breakfast can Bill have?

# The multiplication rule

- If the 1st step of an operation can be performed $n_1$ ways
- And the 2nd step can be performed $n_2$ ways
- …
- And the $k$th step can be performed $n_k$ ways
- Then the operation can be performed $n_1 n_2 \cdot \cdot \cdot n_k$ ways

# Using the multiplication rule for selecting a PIN

- Number of 4 digit PINs of (0,1,2,.)
  – with repetition allowed = 4 • 4 • 4 • 4 = 256
  – with no repetition allowed = 4 • 3 • 2 • 1 = 24
- Extra rules :
  – . (the period) can't be first or last
  – 0 can't be first
    - with repetition allowed = 2 • 4 • 4 • 3
    - without repetition allowed = 2 • 2 • 2 • 1

      (first column, then last column, then middle two)

# Permutations

- Number of ways to arrange $n$ different objects

- Pick first object $n$ ways
- Pick second object $n$-1 ways
- Pick third object $n$-2 ways
- Etc.
- Pick $n$th object 1 way

$$n(n-1)(n-2)\ldots 1 = n!$$

# *r*-Permutations

- Number of ways to arrange *r* different objects out of *n*

- Pick first object *n* ways
- Pick second object *n*-1 ways
- Pick third object *n*-2 ways
- Etc.
- Pick *r*th object *n*-*r*+1 ways

$$n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

# Combinations

- Problem: Choose *r* objects out of *n* (order does not matter).
- Solution: First choose *r* objects out of *n* (order does matter). Then divide by number of orderings of *r* objects.

$$\left[\binom{n}{r} = \frac{n!}{(n-r)!\,r!}\right]$$

# Permutations with Indistinguishable Items I

- Example:
  - Assume you have a set of 15 beads:
    - 6 green
    - 4 orange
    - 3 red
    - 2 black
- How many permutations?
    - Select positions of the green ones, then the orange ones, then the red ones, then the black ones.

$$\binom{15}{6}\binom{9}{4}\binom{5}{3}\binom{2}{2} = \frac{15!}{6!4!3!2!}$$

# Permutations with Indistinguishable Items II

- Example:
  - Assume you have a set of 15 beads:
    - 6 green
    - 4 orange
    - 3 red
    - 2 black
- How many permutations?
    - Take all permutations. Divide by the number of permutations of the green ones, then the orange ones, then the red ones, then the black ones.

$$\frac{15!}{6!4!3!2!}$$

# Permutations with Indistinguishable Items

- Example: Permutations of “revere”

$$\frac{6!}{3!2!} = \frac{720}{6 \cdot 4} = 30$$

# Combinations with repetition

- How many combinations of 20 A's, B's, and C's can be made with unlimited repetition allowed?

- Examples: 10 A’s, 7 B’s, 3 C’s;
  20 A’s, 0 B’s, 0 C’s;
  14 A’s, 0 B’s, 6 C’s.

Reformulate as how may nonnegative solutions to

$$x_1 + x_2 + x_3 = 20$$

# Generalize

-  The number of nonnegative integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r$$

-  The number of selections, with repetition, of size *r* from a collection of size *n*.

-  The number of ways *r* identical objects can be distributed among *n* distinct containers.

Solve in class

# Choosing $r$ elements out of $n$ elements

| | order matters | order doesn't matter |
|---|---|---|
| repetition allowed | $\underbrace{n \times \cdots \times n}_{r\ \text{times}} = n^r$ | $\binom{n+r-1}{r}$ |
| repetition not allowed | $P(n,r) = \frac{n!}{(n-r)!}$ | $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ |

# Where the multiplication rule doesn't work

- People= {Alice, Bob, Carolyn, Dan}
- Need to be appointed as president, vice-president, and treasurer, and nobody can hold more than one office
  - how many ways can it be done with no restrictions?
  - how many ways can it be done if Alice doesn't want to be president?
  - how many ways can it be done if Alice doesn't want to be president, and only Bob and Dan are willing to be vice-president?

# Harder examples of selecting representatives

Candidates= {Azar, Barack, Clinton, Dan, Erin, Fred}

1. Select two, with no restrictions
2. Select two, assuming that Azar and Dan must stay together
3. Select three, with no restrictions
4. Select three, assuming that Azar and Dan must stay together
5. Select three, assuming that Barack and Clinton refuse to serve together

# Properties of combinations and their proofs

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

$$\binom{n}{1} = n \qquad \binom{n}{n-1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2} \qquad \binom{n}{n-2} = \frac{n(n-1)}{2}$$

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$$\binom{n}{r} = \binom{n}{n-r}$$

# A Combinatorial Identity

- How many subsets are there of {1,2, …, *n*}?
- Solution I:

  1 in or out, 2 in or out, …, *n* in or out: Hence $2^n$
- Solution II:

  Can CHOOSE set with

  0 elements, or 1 element, or …, or *n* elements:

  Hence $$\sum_{i=0}^{n} \binom{n}{i}$$

Hence $$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

# The binomial theorem

$$(x+y) = x+y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = (x+y)^2(x+y) = (x^2 + 2xy + y^2)(x+y) =$$
$$x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = (x+y)^3(x+y) = (x^3 + 3x^2y + 3xy^2 + y^3)(x+y) =$$
$$x^4 + 3x^3y + 3x^2y^2 + xy^3 + x^3y + 3x^2y^2 + 3xy^3 + y^4 =$$
$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

# Different types of members

{Alice, Bob, Carol, Dan, Erin, Fred, George, Harry}

Suppose Alice, Carol, and Erin are MATH majors, and the rest are CS majors.

- 8 people in the set: 3 MATHs & 5 CSs
  - make a 5-member team of 2 MATHs and 3 CSs
  - make a 5-member team that has only one MATH
  - make a 5-member team that has no MATHs
  - make a 5-member team that has at least one MATH

# Probability

-  The likelihood of a specific event.
-  Sample space = set of all possible outcomes
-  Event = subset of sample space
-  Equal probability formula:
  - given a finite sample space S where all outcomes are equally likely
  - select an event E from the sample space S
  - the probability of event E from sample space S:

$$P(E) = \frac{|E|}{|S|}$$

# Examples of Sample Spaces

- Two coins
  - sample space = {(H,H), (H,T), (T,H), (T,T)}
- Cards
  - values: 2,3,4,5,6,7,8,9,10,J,Q,K,A
  - suits: D(♦), H(♥), S(♠), C(♣)
- Dice
  - sample space

    {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),
    (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
    …
    (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

# Probabilities with PINs

- Number of four letter PINs of {a,b,c,d}
  - with repetition allowed = $4 \bullet 4 \bullet 4 \bullet 4 = 256$
  - with no repetition allowed = $4 \bullet 3 \bullet 2 \bullet 1 = 24$

- What is the probability that your 4 digit PIN has no repeated digits?

- What is the probability that your 4 digit PIN does have repeated digits?

  Tree method: $4(1\bullet4\bullet4 + 3(2\bullet4 + 2\bullet3))$

# Probability of Poker Hands

- Straight Flush
- Four of a kind
- Full house
- Flush
- Straight
- Three of a kind
- Two pairs
- Pair
- Nothing

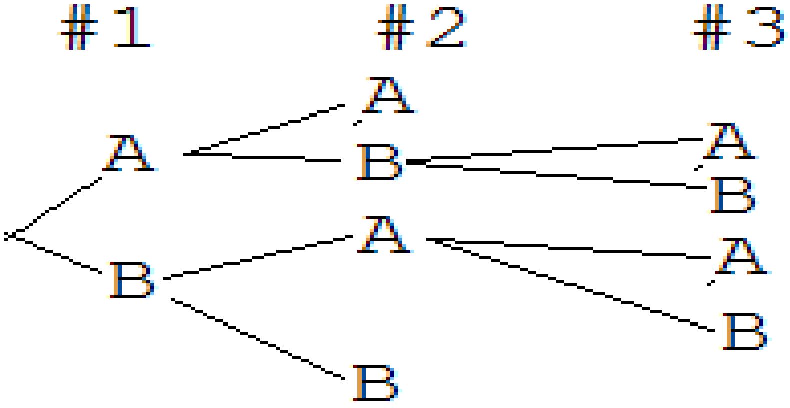
Solve in class

# Multi-level probability

- If a coin is tossed once, the probability of head = ½
- If it's tossed 5 times
  - the probability of all heads: $$\frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} = \frac{1}{2^5}$$
  - the probability of exactly 4 heads: $$\frac{5}{2^5}$$
- This is because the coin tosses are all **independent** events

# Tournament play

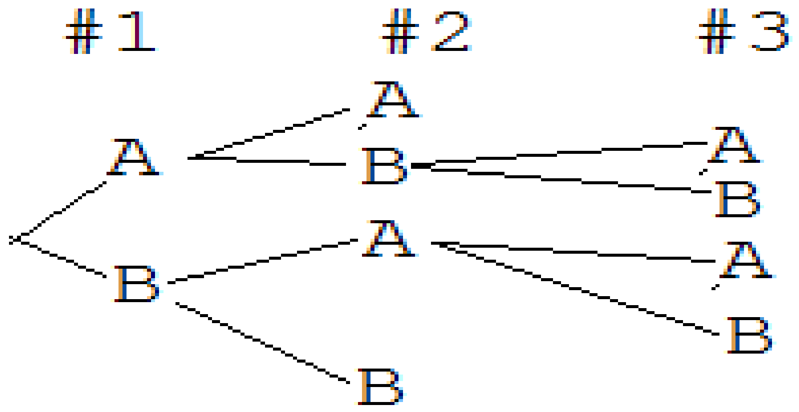
- Team A and Team B compete in a “best of 3” tournament
- They each have an equal likelihood of winning each game




  - Do the leaves add up to 1?
  - Do they always have to play 3 games?
  - What's the probability the tournament finishes in 2 games?
  - Do A and B have an equal chance of winning?

# What if A wins each game with prob 2/3?

- Each line for A must have a 2/3
- Each line for B must have a 1/3




  - How likely is A to win the tournament?
  - How likely is B to win the tournament?
  - What is the probability the tournament finishes in two games?