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Fall 2014 CMSC250/250H Midterm II

Circle Your Section!			
0101 (10am: 3120, Ladan)	0102 (11am: 3120, Ladan)	0103 (Noon: 3120, Peter)	
0201 (2pm: 3120, Yi)	0202 (10am: 1121, Vikas)	0203 (11am: 1121, Vikas)	0204 (9am: 2117, Karthik)
0301 (9am: 3120, Huijing)	0302 (8am: 3120, Huijing)	0303 (1pm: 3120, Yi)	250H (10am: 2117, Peter)

Do not open this exam until you are told. Read these instructions:

1. **Write your name legibly on the top of each page of this exam.** Credit will not be given for lost/missing pages.
2. **CIRCLE YOUR SECTION**
3. **This is a closed book exam.** No calculators, notes, or other aids are allowed. If you have a question during the exam, please raise your hand.
4. You must turn in your exam **immediately** when time is called at the end.
5. In order to be eligible for as much partial credit as possible, show all of your work for each problem, write legibly, and clearly indicate your answers. **Credit will not be given for illegible answers.**
6. After the last page there is paper for scratch work. If you need extra scratch paper after you have filled these areas up, please raise your hand. Scratch paper must be turned in with your exam, with your name and ID number written on it, but scratch paper will not be graded. You may also use the scratch paper as extra space for answers, but you must cross reference it.
7. You may not give or receive any unauthorized assistance on this examination.

Question	1	2	3	4	5	6	7
Points	25	10	10	10	15	10	20
Score							

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1. (25 points) No partial credit will be given on this section. You do not need to show your work.

(a) (4 points) Simplify: $\frac{(n+1)!}{(n-2)!}$.

Solution: $(n+1)(n)(n-1)$

(b) (4 points) Write down an element of the set $(Z \times R) - (Z \times Q)$.

Solution: $(2, \pi)$

(c) (4 points) Evaluate: $\prod_{i=2}^{4}(i-1)^2$. Your answer should be a single number.

Solution:

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(d) (4 points) Give a counter-example that shows this is not a bijection: $f(x) = x^2 + 2$ where $f : R \to R$.

Solution: $f(-1) = f(1)$. Alternately, $y = 0$ has no pre-image.

(e) (4 points) Write the nth term of this sequence in closed form: $0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots$

Solution:

$$a_n = \frac{(-1)^{n+1}n}{n+1}$$

or

$$a_n = \frac{(-1)^n(n-1)}{n}$$

(f) (5 points) What is the smallest positive number that is congruent to -3 mod 11?

Solution:

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2. (10 points)

   (a) **Prove that the interval $A = [1, 3]$ has the same cardinality as $B = [1, 5]$ by writing down a bijection from A to B.** Do NOT prove that your function is a bijection.

   Solution:
   $$f(a) = 2(a-1) + 1 = 2a - 1$$

   (b) Consider the following infinite set:
   $$A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$$

   Prove the set A has the same cardinality as the integers by writing down a bijection from A onto the Z. Do NOT prove that your function is a bijection. Your function must take an element of A as an argument, and return an integer. Your function must be in closed form (no cases!). You may use the floor and ceiling functions.

   Solution:
   $$f(a) = (-1)^{1/a} \left\lfloor \frac{1}{2a} \right\rfloor$$

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3. (10 points) Prove the following using induction:

$$\sum_{i=0}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

Write only one step per line, and place a justification to the right of each step (ex: "By algebra" or "by induction hypothesis").

(a) Base Case:

Solution:
$n = 0$
Sum: $\sum_{i=0}^{0} i(i+1) = 0 \cdot 1 = 0$
Formula: $\frac{n(1)(4)}{6} = 0$

(b) Inductive Hypothesis:

Solution: Assume for some n that

$$\sum_{i=0}^{n-1} i(i+1) = \frac{(n-1)(n)(2(n-1)+4)}{6} = \frac{(n-1)(n)(2n+2)}{6}$$

(c) Inductive Step (label where you use the induction hypothesis):

Solution:

$$\begin{aligned}
\sum_{i=0}^{n} i(i+1) &= n(n+1) + \sum_{i=0}^{n-1} i(i+1) && \text{Sum Splitting} \\
&= n(n+1) + \frac{(n-1)(n)(2n+2)}{6} && \text{By IH} \\
&= \frac{6n(n+1)}{6} + \frac{2(n-1)(n)(n+1)}{6} && \text{By Algebra} \\
&= \frac{6n(n+1) + 2(n-1)(n)(n+1)}{6} && \text{By Algebra} \\
&= \frac{(2(n-1)+6)(n)(n+1)}{6} && \text{By Algebra} \\
&= \frac{(n+2)(n)(n+1)}{3} && \text{By Algebra}
\end{aligned}$$

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4. (10 points) Consider the following sequence:

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 + 2a_0 = 3 \\
a_2 &= 1 + 2a_0 + 2a_1 = 9 \\
a_3 &= 1 + 2a_0 + 2a_1 + 2a_2 = 27 \\
&\vdots \\
a_n &= 1 + \sum_{i=0}^{n-1} 2a_i
\end{aligned}
$$

Using WEAK induction, prove that $a_n = 3^n$ for all $n \geq 0$. Write only one step per line, and place a justification to the right of each step (ex: "By algebra" or "by induction hypothesis").

(a) Base Case:

Solution: $n = 0$
Term: $a_0 = 1$
Formula $= 3^0 = 1$

(b) Inductive Hypothesis:

Solution: Assume for some n that

$$a_n = 3^n.$$

(c) Inductive Step (label where you use the induction hypothesis):

Solution:

$$
\begin{aligned}
a_{n+1} &= 1 + \sum_{i=0}^{n} 2a_i && \text{By Definition} \\
&= 1 + 2a_n + \sum_{i=0}^{n-1} 2a_i && \text{Sum splitting} \\
&= 2a_n + \left(1 + \sum_{i=0}^{n-1} 2a_i\right) && \text{Algebra} \\
&= 2a_n + a_n && \text{Definition of } a_n \\
&= 2 \cdot 3^n + 3^n && \text{IH} \\
&= 3 \cdot 3^n && \text{By Algebra} \\
&= 3^{n+1} && \text{By Algebra}
\end{aligned}
$$

5. (15 points) (a) **Prove:**

$$B \cap A \subseteq (A \cup B) - \bar{B}.$$

Your proof must use element chasing. Your proof MUST be in two-column format, with one step per line, and a justification/reason for that step on the left. You may NOT use the set identities handout.

Solution: Choose some $x \in B \cap A$.

$x \in B$ and $x \in A$	definition of intersection
$x \in A \cup B$	definition of union
$x \notin \bar{B}$	complement
$x \in (A \cup B) - \bar{B}$	definition of difference/complement

(b) **For two sets A and B, prove:**

$$B = (A \cup B) - (A - B).$$

Your proof MUST use the set identities (see handout). Your proof MUST be in two-column format. You may NOT use element chasing.

Solution: Set Algebra:

$(A \cup B) - (A - B) = (A \cup B) \cap \overline{(A - B)}$	Definition
$= (A \cup B) \cap \overline{(A \cap \bar{B})}$	Definition
$= (A \cup B) \cap (\bar{A} \cup B)$	De Morgan's
$= (A \cap \bar{A}) \cup B$	Distributive Law
$= \varnothing \cup B$	Intersection with complement
$= B$	Union with empty set

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6. (10 points) **Prove the following statement using modular arithmetic and cases:**

If $5|n^3$, then $5|n$.

You may NOT use the unique factorization theorem (also called fundamental theorem of arithmetic).

Solution:

We will prove the contrapositive: $5 \not| n \to 5 \not| n^3$.

Assume that $5 \not| n$. Then $n \not\equiv 0 \mod 5$. There are 4 cases:

$$\begin{aligned}
&\text{Case 1: } n \equiv 1 \mod 5, \text{ then } n^3 \equiv 1 \mod 5\\
&\text{Case 2: } n \equiv 2 \mod 5, \text{ then } n^3 = 8 \equiv 3 \mod 5\\
&\text{Case 3: } n \equiv 3 \mod 5, \text{ then } n^3 = 27 \equiv 2 \mod 5\\
&\text{Case 4: } n \equiv 4 \mod 5, \text{ then } n^3 = 64 \equiv 4 \mod 5
\end{aligned}$$

In all cases where $5 \not| n$, $5 \not| n^3$. This proves that $5 \not| n \to 5 \not| n^3$, and thus by contraposition $5|n^3 \to 5|n$.

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7. (20 points) This problem concerns the CUBE root of 5.

   (a) **Prove that $\sqrt[3]{5}$ is irrational. You MUST use the result from question 6. You may NOT use the fundamental theorem of arithmetic (also called the unique factorization theorem).**

   Solution: Assume $\sqrt[3]{5} = p/q$ is a fraction in LOWEST TERMS. Then

$\sqrt[3]{5} = p/q$	By Assumption
$5 = p^3/q^3$	By algebra
$5q^3 = p^3$	By algebra
$5\|p^3$	By Definition of "divides"
$5\|p$	Result of part a
$p = 5n$ for some n	definition of "divides"
$5q^3 = 125n^3$	By algebra
$q^3 = 5(5n^3)$	By algebra
$5\|q$	Definition of "divides"

   Since $5|p$ and $5|q$, the fraction p/q cannot be in lowest terms, which is a contradiction. Therefore, the assumption is false, and $\sqrt[3]{5}$ is irrational.

   (b) **Prove that $\sqrt[3]{5}$ is irrational again. This time you must NOT use the result from question 6, and you MUST use the fundamental theorem of arithmetic (also called the unique factorization theorem).**

   Solution: Assume $\sqrt[3]{5} = p/q$ is a fraction in LOWEST TERMS. Then

$\sqrt[3]{5} = p/q$	By Assumption
$5 = p^3/q^3$	By algebra
$5q^3 = p^3$	By algebra
$p = 5^{n_5} p_1^{n_1} p_2^{n_2} \cdots p_K^{n_K}$, p_i prime	FTA/UFT
$p^3 = 5^{3n_5} p_1^{3n_1} p_2^{3n_2} \cdots p_K^{3n_K}$	Algebra
$q = 5^{m_5} q_1^{m_1} q_2^{m_2} \cdots q_L^{m_L}$, q_i prime	FTA/UFT
$5q^3 = 5^{3m_5+1} q_1^{3m_1} q_2^{3m_2} \cdots p_L^{3m_L}$	Algebra

   The number of factors of 5 in p^3 is divisible by 3. The number of factors of 5 in $5q^3$ is NOT divisible by 3. By the UFT/FTA these quantities cannot be equal, which is a contradiction.

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SCRATCH PAPER

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