Functions

Function terminology

- A *function* maps elements of one set (called the *domain)* to another set (called the *codomain)*.
- *Domain*: the set which contains the values to which the function is applied
- *Codomain*: the set which contains the ***possible*** values (results) of the function
- *Image*: the set of ***actual*** values produced when applying the function to the values of the domain

Formally the image of f is $\{y \in Y \mid (\exists x \in X)[f(x) = y]\}$

More function terminology

- $f: X \rightarrow Y$
  - f is the function name
  - X is the domain
  - Y is the co-domain
  - $f(x) = y$ means f maps x to y

Formal definitions

- Image of function (in class)
- Inverse image of a point (in class)
- f=g if
  - f and g have the same domain
  - f and g have the same codomain
  - For all x (in domain), $f(x)=g(x)$

Types of functions

- $F: X \to Y$ is *one-to-one* (or *injective*) if
  $(\forall x_1, x_2 \in X)[F(x_1) = F(x_2) \to x_1 = x_2]$, or alternatively
  $(\forall x_1, x_2 \in X)[x_1 \neq x_2 \to F(x_1) \neq F(x_2)]$
- $F: X \to Y$ is *onto* (or *surjective*) if
  $(\forall y \in Y)(\exists x \in X)[F(x) = y]$
- $F: X \to Y$ is ***not*** *one-to-one* if
  $(\exists x_1, x_2 \in X)[(x_1 \neq x_2) \wedge (F(x_1) = F(x_2))]$
- $F: X \to Y$ is ***not*** *onto* if
  $(\exists y \in Y)(\forall x \in X)[F(x) \neq y]$

Proving functions one-to-one and onto

$f(x) = 3x - 4, \quad f(x) = x^2+2$

- $f: R \rightarrow R$
- $f: R \rightarrow Z$
- $f: Z \rightarrow R$
- $f: Z \rightarrow Z$

Solve in class

One-to-one correspondence or bijection

$F: X \to Y$ is *bijective* iff $F: X \to Y$ is one-to-one and onto

If $F: X \to Y$ is bijective then it has an inverse function

$$(\exists F^{-1})[Y \to X]$$

$$(\forall x \in X)[F(x) = y \to F^{-1}(y) = x]$$

$$(\forall y \in Y)[F^{-1}(y) = x \to F(x) = y]$$

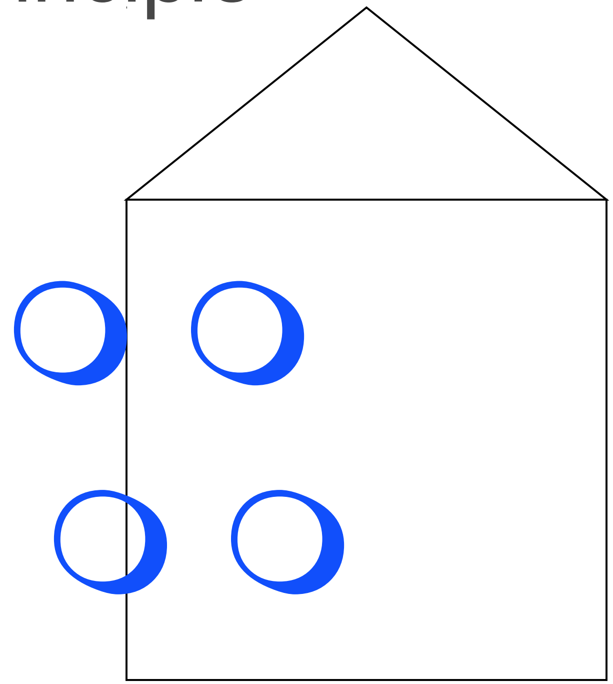
Proving something is a bijection

$$F(x) = 5x + \tfrac{1}{2}$$

- $f: R \rightarrow R$
- $f: R \rightarrow Z$
- $f: Z \rightarrow R$
- $f: Z \rightarrow Z$

Solve in class

The pigeonhole principle



- Basic form:

  A function from one finite set to a smaller finite set cannot be one-to-one; there must be at least two elements in the domain that have the same image in the codomain.

Examples

- Using this class as the domain:
  – must two people share a birth month?
  – must two people share a birthday?

- Let $A = \{1,2,3,4,5,6,7,8\}$
  – if I select 5 different integers at random from this set, must two of the numbers sum exactly to 9?
  – if I select 4 integers?

- There exist two people in New York City who have the same number of hairs on their heads.

- There exist two subsets of $\{1,\ldots,10\}$ with three elements that sum to the same value.

Generalized Pigeonhole Principle

- The generalized pigeonhole principle:
  - If X,Y are finite, and $|X|>k|Y|$, and $f:X \rightarrow Y$ then there is some $y \in Y$ such that y is the image of at least $k+1$ distinct elements of X.

Examples

- Using the generalized form:
  - assume 50 people in the room, how many must share the same birth month?
  - $|A|=5$, $|B|=3$ F: $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$

    how many elements of $\mathcal{P}(A)$ must map to a single element of $\mathcal{P}(B)$?

Composition on finite sets- example

- Example

  $X = \{1,2,3\}$, $Y_1 = \{a,b,c,d\}$, $Y = \{a,b,c,d,e\}$, $Z = \{x,y,z\}$

$f(1) = c$	$g(a) = y$	$g \circ f(1) = g(f(1)) = z$
$f(2) = b$	$g(b) = y$	$g \circ f(2) = g(f(2)) = y$
$f(3) = a$	$g(c) = z$	$g \circ f(3) = g(f(3)) = y$
	$g(d) = x$	
	$g(e) = x$	

Composition for infinite sets- example

$f: Z \rightarrow Z \quad f(n) = n + 1$

$g: Z \rightarrow Z \quad g(n) = n^2$

$g \circ f(n) = g(f(n)) = g(n+1) = (n+1)^2$

$f \circ g(n) = f(g(n)) = f(n^2) = n^2 + 1$

Note: $g \circ f \neq f \circ g$

Identity function

- i_X the identity function for the domain X

  $i_X : X \to X \quad (\forall x \in X)\ [i_X(x) = x]$

- i_Y the identity function for the domain Y

  $i_Y : Y \to Y \quad (\forall y \in Y)\ [i_Y(y) = y]$

- composition with the identity functions

Composition with inverse

- Recall: if f is a bijection then f^{-1} exists.
- Let $f: X \to Y$ be a bijection.
- What is $f \circ f^{-1}$?
- What is $f^{-1} \circ f$?

One-to-one in composition

- If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one, then $g \circ f: X \to Z$ is one-to-one.

- If $f: X \to Y$ and $g: Y \to Z$ are both onto, then $g \circ f: X \to Z$ is onto.

Cardinality

- Sets A and B have the *same cardinality* iff there is a one-to-one correspondence from A to B.

- We denote A and B have the same cardinality by $|A| = |B|$.

Countable sets

- A set S is called *countably infinite* iff |S| = |**N**|.
- A set is called *countable* iff it is finite or countably infinite.
- A set which is not countable is called *uncountable*.

Countable Sets?

Which of these sets are countably infinite?

- N
- N^{even}
- Z
- Q^{+}
- Q

Real numbers

- We'll take just a part of this infinite set

- Reals between 0 and 1 (noninclusive)

  $X = \{x \in \mathrm{R} \mid 0 < x < 1\}$

- All elements of X can be written as

  $0.a_1a_2a_3\ldots\ a_n\ldots$

Cantor's proof

- Assume the set $X = \{x \in \mathrm{R} \mid 0 < x < 1\}$ is countable
- Then the elements in the set can be listed

  $0.a_{11}a_{12}a_{13}a_{14}\ldots a_{1n}\ldots$

  $0.a_{21}a_{22}a_{23}a_{24}\ldots a_{2n}\ldots$

  $0.a_{31}a_{32}a_{33}a_{34}\ldots a_{3n}\ldots$

  … … … …
- Select the digits on the diagonal
- Build a number d, such that d differs in its n^{th} position from the n^{th} number in the list

$$d_n = \begin{cases} 1 \text{ if } & a_{nn} \neq 1 \\ 2 \text{ if } & a_{nn} = 1 \end{cases}$$

All reals

Cardinality($\{x \in R \mid 0 < x < 1\}$) = Cardinality(R)