QUANTIFIERS

PROPOSITIONS VS PREDICATES

Propositions: statements with truth value

$$1 + 2 > 4$$

Bill and Ted had an excellent adventure

Predicates: statements with variables

$$x + 2 > 4$$

X and Y had an excellent adventure

PREDICATE VARIABLES

Compound Propositions:

$$p \wedge q \rightarrow r$$

Logical variables have truth value

Predicate Variables

$$x + 2 > 4$$

X and Y had an excellent adventure

Variables range over a set - "Domain of discourse"

PROPOSITIONAL FUNCTIONS

P(X,Y) = X and Y had an excellent adventure

- NOT a proposition, a predicate
- Also called propositional function
- Map domain elements onto propositions

Propositions

P(Bill, Ted) = Bill and Ted had an excellent adventure **T**

P(JayZ, Solange) = JayZ and Solange had an excellent adventure **F**

UNIVERSAL QUANTIFIERS

$$\forall$$ "For all"

$\forall x, P(x)$ "For all x, P(x) is true"

Examples

$$\forall x, x + 1 > x$$

$$\forall x, x < 1$$

$$\forall x, x \text{ and Ted had an excellent adventure}$$

Truth value depends on domain of discourse

EXAMPLE 1

$$Q(x) = x < 2$$

Is this statement true? $\forall x, Q(x)$

If domain is $\mathbb{R}$: **F**

Counterexample: $x = 2$

If domain is integers less than 2: **T**

EXAMPLE 2

$$Q(x) = x < 2$$

Is this statement true? $\forall x, Q(x)$

If domain is integers between -2 and 1 (inclusive)?

$$Q(-2) \wedge Q(-1) \wedge Q(0) \wedge Q(1)$$ **T**

If domain is all positive integers?

$$Q(1) \wedge Q(2) \wedge Q(3) \wedge Q(4) \wedge Q(5) \cdots$$ **F**

EXISTENTIAL QUANTIFIERS

$$\exists$$ "There Exists"

$\exists x, P(x)$ "The exists an x such that P(x) is true"

"There is at least one x with P(x) true"

"For some x, P(x) holds"

EXAMPLE

$$Q(x) = x < 2$$

Is this statement true? $\exists x, Q(x)$

If domain is integers between -2 and 1 (inclusive)?

$$Q(-2) \vee Q(-1) \vee Q(0) \vee Q(1)$$ **T**

If domain is all positive integers?

$$Q(1) \vee Q(2) \vee Q(3) \vee Q(4) \vee Q(5) \cdots$$ **T**

DE MORGAN'S LAW (AGAIN)

Propositional	Quantifiers
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg\forall x, P(x) \equiv \exists x, \neg P(x)$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg\exists x, P(x) \equiv \forall x, \neg P(x)$

First law: Existence of counter-example

Second law: Un-satisfiability

ENGLISH TO LOGIC

"Every student in this class has taken calculus"

Domain: Students in this class

Define predicate: C(x)="x took calculus"

$$\forall x, C(x)$$

Domain: All students in the world
Define predicate: S(x)="x is in this class"

$$\forall x, S(x) \to C(x)$$

MORE ENGLISH TO LOGIC

M(x): "x went to Mexico"

D(x): "x went to Denmark"

S(x): "x is a student"

Domain: All people

"Some student has visited Mexico"

"No student has visited Mexico"

"Every student visited Mexico but not Denmark"

"Every student visited Mexico but not all visited Denmark"

NESTED QUANTIFIERS

Write in words:

$$\forall x \exists y, x + y = 0$$

"For every x, there exists a y with x+y=0"

"Every real number has an additive inverse"

$$\forall x \forall y, (x > 0) \wedge (y < 0) \rightarrow xy < 0$$

"For all x, for all y, if x>0 and y<0 then xy<0"

"The product of a positive number and a negative number is negative"

DOES ORDER MATTER?

$$\forall x \forall y, x^2y^2 \geq 0$$

For all x, is holds for all y that $x^2y^2 \geq 0$ **T**

Order doesn't matter

For all x and y, $x^2y^2 \geq 0$ **T**

$$\forall x \exists y, x + y = 0$$

For all x, we can find a y such that x+y=0 **T**

$$\exists y \forall x, x + y = 0$$

We can find a y such that for all x, x+y=0 **F**

EXAMPLES: ENGLISH TO LOGIC

"The sum of two positive integers is positive"

"Every non-zero real number has a multiplicative inverse"

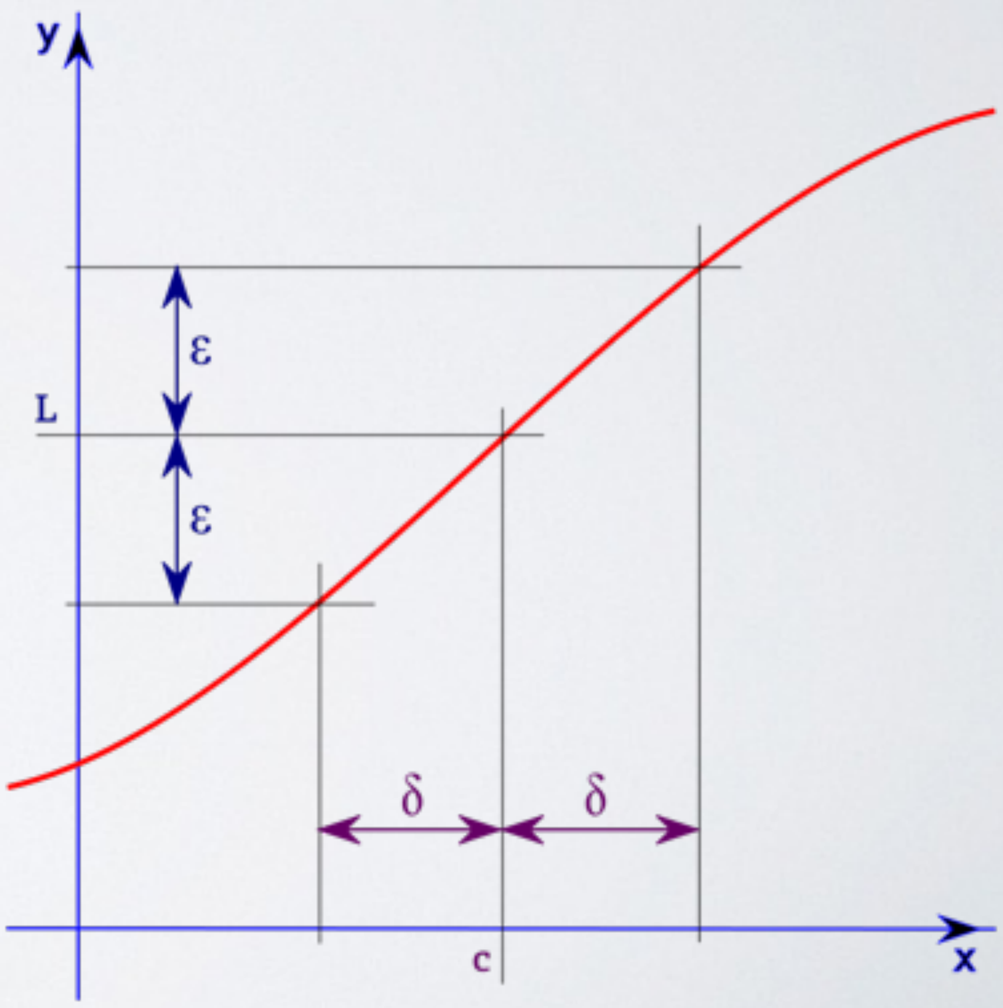
Limits:

$$\lim_{x \to a} f(x) = L$$

For every positive epsilon, there exists a positive delta such that

$$|f(x) - L| < \epsilon$$

whenever $|x - a| < \delta$.



EXAMPLES: LOGIC TO ENGLISH

C(x): x has a computer
F(x,y): x and y are friends

Domain: all students in the class

$$\forall x, C(x) \lor \exists y, (C(y) \land F(x, y))$$

$$\exists x, \forall y, \forall z, F(x, y) \land F(x, z) \land (y \neq z) \rightarrow \neg F(y, z)$$

NEGATING NESTED QUANTIFIERS

Prove using De Morgan's Laws:

$$\neg\forall x\exists y\forall z P(x) \wedge Q(y,z)$$
$$= \exists x\forall y\exists z\neg P(x) \vee \neg Q(y,z))$$