Name & Directory ID: ____________________

| Circle Your Section! | | | |
|---|---|---|---|
| 0101 (10am: 3120, Ladan) | 0102 (11am: 3120, Ladan) | 0103 (Noon: 3120, Peter) | |
| 0201 (2pm: 3120, Yi) | 0202 (10am: 1121, Vikas) | 0203 (11am: 1121, Vikas) | 0204 (9am: 2117, Karthik) |
| 0301 (9am: 3120, Huijing) | 0302 (8am: 3120, Huijing) | 0303 (1pm: 3120, Yi) | 250H (10am: 2117, Peter) |

In Problems (1-3) you will derive logical properties using the Laws of Logic provided on the last page of this assignment. You must do this by stating, to the right of each line, what rule is used (as done in the book).

1. (25 points) Show that the following statements are equivalent:

$$(p \vee q) \rightarrow r \quad \text{and} \quad (p \rightarrow r) \wedge (q \rightarrow r)$$

(a) Construct a truth-table.

**Solution:**

| $p$ | $q$ | $r$ | $(p \vee q)$ | $(p \vee q) \rightarrow r$ | $p \rightarrow r$ | $q \rightarrow r$ | $(p \rightarrow r) \wedge (q \rightarrow r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | F |
| T | F | T | T | T | T | T | T |
| T | F | F | T | F | F | T | F |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | T | F | F |
| F | F | T | F | T | T | T | T |
| F | F | F | T | T | T | T | T |

(b) Derive one from the other using the logical properties.

**Solution:**

$$\begin{aligned}
& (p \vee q) \rightarrow r & \\
\equiv\ & \neg(p \vee q) \vee r & \text{by the definition of implication} \\
\equiv\ & (\neg p \wedge \neg q) \vee r & \text{by deMorgan's laws} \\
\equiv\ & (\neg p \vee r) \wedge (\neg q \vee r) & \text{by the distributive property} \\
\equiv\ & (p \rightarrow r) \wedge (q \rightarrow r) & \text{by the definition of implication}
\end{aligned}$$

2. (25 points) Use the logical properties to show that the following statements are equivalent (by deriving one from the other):

$$(p \rightarrow q) \wedge (p \rightarrow r) \quad \text{and} \quad (p \rightarrow (q \wedge r))$$

**Solution:**

$$\begin{array}{lll} & (p \rightarrow q) \wedge (p \rightarrow r) & \\ \equiv & (\neg p \vee q) \wedge (\neg p \vee r) & \text{by the definition of implication} \\ \equiv & \neg p \vee (q \wedge r) & \text{by the distributive property} \\ \equiv & p \rightarrow (q \wedge r) & \text{by the definition of implication} \end{array}$$

3. (25 points) Use the logical properties to derive that the following statement is a tautology:

$$(p \rightarrow q) \rightarrow ((p \vee r) \rightarrow (q \vee r))$$

**Solution:**

$$\begin{array}{lll} & (p \rightarrow q) \rightarrow [(p \vee r) \rightarrow (q \vee r)] & \\ \equiv & \neg(p \rightarrow q) \vee [(p \vee r) \rightarrow (q \vee r)] & \text{by the definition of implication} \\ \equiv & \neg(p \rightarrow q) \vee [\neg(p \vee r) \vee (q \vee r)] & \text{by the definition of implication} \\ \equiv & \neg(p \rightarrow q) \vee [(\neg p \wedge \neg r) \vee (q \vee r)] & \text{by deMorgan's laws} \\ \equiv & \neg(p \rightarrow q) \vee [(\neg p \vee (q \vee r)) \wedge (\neg r \vee (q \vee r))] & \text{by the distributive property} \\ \equiv & \neg(p \rightarrow q) \vee [((\neg p \vee q) \vee r)) \wedge (q \vee (\neg r \vee r))] & \text{rearranging terms} \\ \equiv & \neg(p \rightarrow q) \vee [((p \rightarrow q) \vee r)) \wedge (q \vee \mathbf{T}))] & \text{by definition of implication and negation law} \\ \equiv & \neg(p \rightarrow q) \vee [((p \rightarrow q) \vee r) \wedge \mathbf{T})] & \text{by universal bounds law} \\ \equiv & \neg(p \rightarrow q) \vee [(p \rightarrow q) \vee r] & \text{by identity law} \\ \equiv & [\neg(p \rightarrow q) \vee (p \rightarrow q)] \vee r & \text{by associative property} \\ \equiv & \mathbf{T} \vee r & \text{by negation law} \\ \equiv & \mathbf{T} & \text{by universal bound law} \end{array}$$

4. (25 points) A *square-free 3-coloring of* $1, \ldots, n$ is an assignment of Red, Blue, or Green to the integers $1, \ldots, n$ so that no two integers exactly a perfect square apart have the same color.

   For example, RBGRBGRBG is a square-free 3-coloring of $1, \ldots, 9$ since no two integers either $1 = 1^2$ or $4 = 2^2$ apart have the same color. In contrast, RBGRBGRBGRBG is not a square-free 3-coloring of $1, \ldots, 12$ since 1 and 10, which are $9 = 3^2$ apart are both Red. Similarly, 2 and 11 are both Blue, and 3 and 12 are both Green.

   We would like to create a satisfiability problem to find a square-free 3-coloring of $1, \ldots, 16$. Let $R(1)$, $B(1)$, and $G(1)$ be Boolean variables (or propositions) representing number 1 is colored Red, Blue, or Green, respectively. Similarly, for $R(2)$, $B(2)$, $G(2)$, ..., $R(16)$, $B(16)$, $G(16)$. It would be tedious to write the entire logical formula. Just write the part of the formula involving variables $R(1)$, $B(1)$, and $G(1)$.

**Solution:**

$$
\begin{array}{rl}
 & (R(1) \vee B(1) \vee G(1)) \\
\wedge & \neg(R(1) \wedge B(1)) \wedge \neg(R(1) \wedge G(1)) \wedge \neg(B(1) \wedge G(1)) \\
\wedge & \neg(R(1) \wedge R(2)) \wedge \neg(R(1) \wedge R(5)) \wedge \neg(R(1) \wedge R(10)) \\
\wedge & \neg(B(1) \wedge B(2)) \wedge \neg(B(1) \wedge B(5)) \wedge \neg(B(1) \wedge B(10)) \\
\wedge & \neg(G(1) \wedge G(2)) \wedge \neg(G(1) \wedge G(5)) \wedge \neg(G(1) \wedge G(10))
\end{array}
$$

Laws of Logic

| Given any statement variables $p$, $q$, and $r$ the following logical equivalences hold: | | |
|---|---|---|
| 1. Commutative laws: | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. Associative laws: | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. Distributive laws: | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. Identity laws: | $p \wedge \mathbf{T} \equiv p$ | $p \vee \mathbf{F} \equiv p$ |
| 5. Negation laws: | $p \vee \neg p \equiv \mathbf{T}$ | $p \wedge \neg p \equiv \mathbf{F}$ |
| 6. Double Negative law: | $\neg(\neg p) \equiv p$ | |
| 7. Idempotent laws: | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. DeMorgan's laws: | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ |
| 9. Universal bounds laws: | $p \vee \mathbf{T} \equiv \mathbf{T}$ | $p \wedge \mathbf{F} \equiv \mathbf{F}$ |
| 10. Absorption laws: | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. Negations of **T** and **F**: | $\neg\mathbf{T} \equiv \mathbf{F}$ | $\neg\mathbf{F} \equiv \mathbf{T}$ |