

Name & UID: _____

Circle Your Section!			
0101 (10am: 3120, Ladan)	0102 (11am: 3120, Ladan)	0103 (Noon: 3120, Peter)	
0201 (2pm: 3120, Yi)	0202 (10am: 1121, Vikas)	0203 (11am: 1121, Vikas)	0204 (9am: 2117, Karthik)
0301 (9am: 3120, Huijing)	0302 (8am: 3120, Huijing)	0303 (1pm: 3120, Yi)	250H (10am: 2117, Peter)

1. (25 points) What must be the case if the sum of any three integers is odd but their product is even?

(a) Form a conjecture about the three numbers.

Solution: Two of the number are even, and the other is odd.

- (b) Prove your conjecture. Use may use the facts that the sum of two evens is even, two odds is even, and an even and odd is odd; and the product of two evens is even, two odds is odd, and an even and odd is even.

Solution: We know that the product of odd numbers is odd. Therefore, if all three numbers are odd, then their product would be odd. Since the product is even, it must be the case that at least one number is even.

Now suppose exactly one number is even and two numbers are odd. Since the sum of two odds is even, and the sum of an even with an even is even, the sum of the three numbers is even. This is a contradiction. Therefore it is not the case that only one number is even.

Suppose there are three even numbers. Then the sum would be even, which is another contradiction.

Since the number of even numbers cannot be zero, one, or three, there must be two even numbers and one odd.

2. (20 points) Prove by contradiction that the sum of a rational number and an irrational number is irrational.

Solution:

Proof. Proof by contradiction. Assume not. Then the sum of a rational number and an irrational number is rational. Let a/b and c/d be two rational numbers, and r , irrational. Then $a/b + r = c/d$, for integers a, b, c, d , where $b, d \neq 0$. We can rearrange terms so that

$$r = c/d - a/b = \frac{cb - da}{db} ,$$

which is rational since the integers are closed under addition and multiplication and $b, d \neq 0$. This is a contradiction because an irrational number now equals a rational number. $\square$

3. (20 points) Prove that the non-zero rationals are closed under division.

Solution: Any two nonzero rationals can be written a/b and c/d where a, b, c, d nonzero integers. Then

$$\frac{a/b}{c/d} = \frac{ad}{bc} ,$$

which is a nonzero rational since the integers are closed under multiplication and $a, b, c, d \neq 0$.

4. (20 points) Prove that there is an irrational number between any two rationals. You may use the following two lemmas: (1) The sum of a rational and an irrational is irrational, and (2) the product of a non-zero rational and an irrational is irrational.

Solution: Let a, b be two rationals. Assume WLOG that $a < b$. Consider the number

$$a + (b - a)\sqrt{2}/2.$$

Since $\sqrt{2}/2 < 1$, we have

$$a < a + (b - a)\sqrt{2}/2 < a + (b - a)(1) = b.$$

Also, because of Lemmas (1) and (2), we know that $a + (b - a)\sqrt{2}/2$ is irrational. We have thus constructed an irrational between the two rationals.

ALTERNATIVE: Find an irrational smaller than $b - a$ (and then add it to a). Find n such that

$$\frac{\sqrt{2}}{2^n} \leq \frac{b - a}{2} \implies 2^n \geq \frac{2\sqrt{2}}{b - a} \implies n \geq \lg \frac{2\sqrt{2}}{b - a}$$

So $n = \left\lceil \lg \frac{2\sqrt{2}}{b - a} \right\rceil$ suffices.

ALTERNATIVE: There are nonconstructive solutions. We do not intend them, but will allow them.

5. (20 points) Prove, using cases, that for any real number x : if $|x - 10| < 3$, then $x > 7$.

Solution: We need to break up the values of x so that stuff inside the absolute value is positive or negative. We have two cases:

Case 1: $x - 10 \geq 0$. In this case, $x \geq 10$, and therefore $x > 7$.

Case 2: $x - 10 < 0$. In this case, $|x - 10| = -x + 10$. We therefore have

$$-x + 10 < 3$$

which simplifies to

$$x > 7.$$