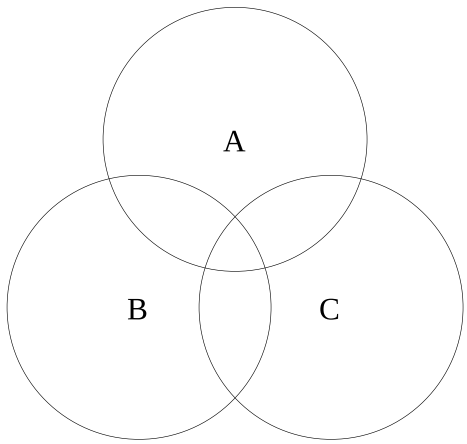
Name & UID: ____________________

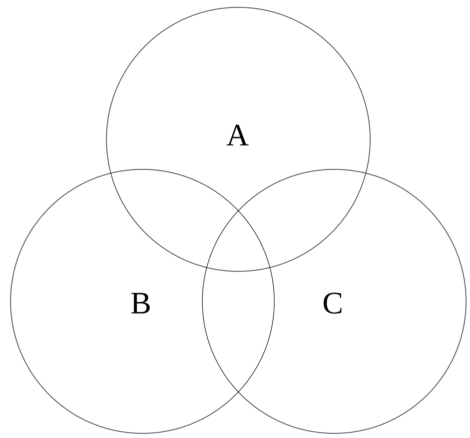
Circle Your Section!			
0101 (10am: 3120, Ladan)	0102 (11am: 3120, Ladan)	0103 (Noon: 3120, Peter)	
0201 (2pm: 3120, Yi)	0202 (10am: 1121, Vikas)	0203 (11am: 1121, Vikas)	0204 (9am: 2117, Karthik)
0301 (9am: 3120, Huijing)	0302 (8am: 3120, Huijing)	0303 (1pm: 3120, Yi)	250H (10am: 2117, Peter)

1. (20 points) Consider the following proposition:

Proposition. *For all sets A, B, and C:* $(A \cap B) - C \subseteq (A \cup B) - C$.

(a) Draw two Venn diagrams, one for $(A \cap B) - C$ and the other for $(A \cup B) - C$.





Solution: OMITTED.

(b) Give an element-chasing proof of the proposition (i.e., show that the left side is a subset of the right side).

Solution:

Proof. Choose a arbitrary in $(A \cap B) - C$, then by the definition of set difference, $a \in A$, and $a \in B$, but $a \notin C$. Because $a \in A$ and $a \in B$, it is certainly in $A \cup B$, by the definition of union, and $a \notin C$. Because, a was arbitrarily chosen in $(A \cap B) - C$, we may conclude that $(A \cap B) - C \subseteq (A \cup B) - C$, which was to be shown. $\square$

2. (20 points) Consider the proposition:

Proposition. *For all sets A and B:* $(A - B) \cap (B - A) = \emptyset$.

(a) Prove the proposition using the set identities (provided at the end of this assignment).

Solution:

$$
\begin{array}{lll}
 & (A - B) \cap (B - A) & \text{Given} \\
\equiv & (A \cap \bar{B}) \cap (B \cap \bar{A}) & \text{by definition of set difference} \\
\equiv & (A \cap \bar{A}) \cap (B \cap \bar{B}) & \text{by associative and commuative laws} \\
\equiv & (\emptyset) \cap (\emptyset) & \text{by intersection with complement} \\
\equiv & \emptyset & \text{by idempotent law}
\end{array}
$$

(b) Give an element-chasing proof of the proposition above. You may use the fact that if X is a set then $X - X = \emptyset$.

Solution: ALTERNATIVE:

Proof.

$$
\begin{array}{lll}
 & x \in (A - B) \cap (B - A) & \\
\Leftrightarrow & (x \in (A - B)) \wedge (x \in (B - A)) & \text{by definition of intersection} \\
\Leftrightarrow & ((x \in A) \wedge (x \notin B)) \wedge ((x \in B) \wedge (x \notin A)) & \text{by definition of set difference} \\
\Leftrightarrow & ((x \in A) \wedge (x \notin A)) \wedge ((x \in B) \wedge (x \notin B)) & \text{by associative and commuative laws} \\
\Leftrightarrow & (x \in (A - A)) \wedge (x \in (B - B)) & \text{by definition of intersection} \\
\Leftrightarrow & (x \in \emptyset) \wedge (x \in \emptyset) & \text{by difference of set with itself} \\
\Leftrightarrow & x \in \emptyset & \text{by idempotent law}
\end{array}
$$

□

ALTERNATIVE:

Proof.

$$
\begin{array}{lll}
 & x \in (A - B) \cap (B - A) & \\
\Leftrightarrow & (x \in (A - B)) \wedge (x \in (B - A)) & \text{by definition of intersection} \\
\Leftrightarrow & ((x \in A) \wedge (x \notin B)) \wedge ((x \in B) \wedge (x \notin A)) & \text{by definition of set difference} \\
\Leftrightarrow & ((x \in A) \wedge (x \notin A)) \wedge ((x \in B) \wedge (x \notin B)) & \text{by associative and commuative laws} \\
\Leftrightarrow & ((x \in A) \wedge \neg(x \in A)) \wedge ((x \in B) \wedge \neg(x \in B)) & \text{by definition of } \notin \\
\Leftrightarrow & (\text{F}) \wedge (\text{F}) & \text{by negation law} \\
\Leftrightarrow & \text{F} & \text{by idempotent law} \\
\Leftrightarrow & x \in \emptyset & \text{vacuously true}
\end{array}
$$

□

3. (15 points) Give an element-chasing proof of the following proposition:

   Proposition. *Given sets A, B and C: if $A - B \subseteq C$, then $A - C \subseteq B$.*

Solution:

Proof. Choose x, arbitrary, in $A - C$, then $x \in A$ and $x \notin C$. To show that $x \in C$, suppose that $a \notin B$, then $x \in A - B$. Because $A - B \subseteq C$, then $x \in C$. But, this contradicts the assertion that $a \notin C$, so we conclude that a must be in B, and thus when $x \in A - C$, then $x \in B$. Because x was chosen arbitrarily, we can conclude that for all $x \in A$, $x \in A - C$ implies that $x \in B$ so $A - C \subseteq B$, which was to be shown. $\square$

4. (12 points) Suppose that $A = \{1, 2\}$ and $B = \{2, 3\}$.

   (a) Display the power set, $\mathcal{P}(A)$.

   Solution:
   $$\{\emptyset, \{1\}, \{2\}, \{1, 2, \}\}$$

   (b) Display the power set, $\mathcal{P}(B)$.

   Solution:
   $$\{\emptyset, \{2\}, \{3, \}, \{2, 3\}\}$$

   (c) Display the power set, $\mathcal{P}(A) \cup \mathcal{P}(B)$.

   Solution:
   $$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, \}, \{2, 3\}\}$$

   (d) Display the power set, $\mathcal{P}(A \cup B)$.

   Solution:
   $$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}.$$

5. (8 points) Given sets $A = \{a, b\}, B = \{2\}$, and $C = \{\mathrm{T}, \mathrm{F}\}$,

   (a) Display $A \times B$.

   Solution:
   $$\{(a, 2), (b, 2)\}$$

   (b) Display $(A \times B) \times C$.

   Solution:
   $$\{((a, 2), \mathrm{T}), ((a, 2), \mathrm{F}), ((b, 2), \mathrm{T}), ((b, 2), \mathrm{F})\}$$

6. (15 points) Prove or disprove that the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, given by the rule $f(x) = (2x-1)/2$, is a bijection.

Solution: ALTERNATIVE:

Proof. To show that f is a bijection we must show (1) that it is injective (one-to-one), and (2) that it is surjective (onto).

To show that f is injective (one-to-one) show that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

$$\frac{2x_1 - 1}{2} = \frac{2x_2 - 1}{2} \implies 2x_1 - 1 = 2x_2 - 1 \implies 2x_1 = 2x_2 \implies x_1 = x_2$$

To show that f is surjective (onto) find the inverse:

$$y = \frac{2x-1}{2} \implies 2y = 2x - 1 \implies 2x = 2y + 1 \implies x = \frac{2y+1}{2}$$

So the inverse function $f^{-1}(y) = \dfrac{2y+1}{2}$ (or $f^{-1}(x) = \dfrac{2x+1}{2}$). So for any value of y there exists a value of x that maps to it by $f(x)$. $\square$

ALTERNATIVE:

Proof. Let A and B be the real numbers $\mathbb{R}$, and let $f\colon A \longrightarrow B$, be defined by the rule

$$f(x) = \frac{(2x-1)}{2}.$$

To show that f is a bijection we need show that there exists an inverse function, $g\colon B \longrightarrow A$, such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.
Define $g\colon B \longrightarrow A$, as $\frac{1}{2}(2x+1)$.
Composing $g \circ f$:

$$g \circ f = \frac{1}{2}\left[2\frac{(2x-1)}{2} + 1\right] = x.$$

Composing $f \circ g$:

$$f \circ g = \frac{2\left(\dfrac{(2x+1)}{2}\right) - 1}{2} = x.$$

$\square$

Because g satisfies the definition of the inverse of f, the function f is bijective.

7. (10 points) Show that if $g\colon A \longrightarrow B$ and $h\colon B \longrightarrow X$ are both injective (one-to-one), then $h \circ g$ is also injective (one-to-one).

Solution:

Proof. We need to show that $h(g(x_1)) = h(g(x_2)) \implies x_1 = x_2$. Assume that $h(g(x_1)) = h(g(x_2)$. Then $g(x_1) = g(x_2)$ since h is one-to-one. Then $x_1 = x_2$ since g is one-to-one. □

Question:	1	2	3	4	5	6	7	Total
Points:	0	0	0	0	0	0	0	0
Score:								