CMSC250 Homework 7 Due: Wednesday, November 5, 2014

Name & UID: ______________________________

Circle Your Section!			
0101 (10am: 3120, Ladan)	0102 (11am: 3120, Ladan)	0103 (Noon: 3120, Peter)	
0201 (2pm: 3120, Yi)	0202 (10am: 1121, Vikas)	0203 (11am: 1121, Vikas)	0204 (9am: 2117, Karthik)
0301 (9am: 3120, Huijing)	0302 (8am: 3120, Huijing)	0303 (1pm: 3120, Yi)	250H (10am: 2117, Peter)

Question:	1	2	3	4	5	6	Total
Points:	10	20	20	30	10	10	100
Score:							

1. (10 points) Show that if $7|n^2$ then $7|n$ (without using the Unique Factorization Theorem).

Solution: We prove contrapositive: if $7 \not| x$ then $7 \not| x^2$.

If $7 \not| x$ then $x \equiv 1, 2, 3, 4, 5, 6 \pmod 7$.
If $x \equiv 1$ then $x^2 \equiv 1^2 \equiv 1 \not\equiv 0$.
If $x \equiv 2$ then $x^2 \equiv 2^2 \equiv 4 \not\equiv 0$.
If $x \equiv 3$ then $x^2 \equiv 3^2 \equiv 9 \equiv 2 \not\equiv 0$.
If $x \equiv 4$ then $x^2 \equiv 4^2 \equiv 16 \equiv 2 \not\equiv 0$.
If $x \equiv 5$ then $x^2 \equiv 5^2 \equiv 25 \equiv 4 \not\equiv 0$.
If $x \equiv 6$ then $x^2 \equiv 6^2 \equiv 36 \equiv 1 \not\equiv 0$.

2. (20 points) Show that $\sqrt{7}$ is irrational.

   (a) Use the Unique Factorization Theorem.

   Solution: Assume $\sqrt{7} = a/b$, so $7b^2 = a^2$. In the factorization of a^2 every prime appears with an even exponent. In the factorization of $7b^2$ every prime appears with an even exponent except for 7 which has an odd exponent. Since $7b^2 = a^2$ this is a contradiction (by UFT).

   (b) Do not use the Unique Factorization Theorem.

   Solution: Assume $\sqrt{7} = a/b$, so $7b^2 = a^2$. We can assume a and b have been reduced so that they have no common divisors.

   It must be the case that $7|a^2$, so by the previous problem $7|a$. Therefore $a = 7m$ for some integer k. By substitution $7b^2 = (7k)^2$. Therefore $b^2 = 7k^2$. It must be the case that $7|b^2$, so by the previous problem $7|b$. Thus, $7|a$ and $7|b$ so a and b have a common factor, which is a contradiction.

3. (20 points) Consider the sequence $w_0, w_1, w_2, \ldots$, where w_i is the number of 1's in the binary representation of i.

   (a) Write down the first thirteen terms of the sequence (i.e., $w_0, w_1, w_2, \ldots, w_{12}$). No justification needed.

   Solution: 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2

   (b) Calculate $\sum_{i=0}^{12} w_i$. No justification needed.

   Solution: 22

4. (30 points) For each sequence, write a simple, closed-form expression for the ith term. HINT: It can sometimes be helpful look at the difference of consecutive terms ($a_{i+1} - a_i$) or the ratio of consecutive terms (a_{i+1}/a_i) to see what is happening.

   (a) $-\frac{1}{2}, \frac{1}{5}, -\frac{1}{8}, \frac{1}{11}, -\frac{1}{14}, \ldots,$

   Solution:
   $$\frac{(-1)^i}{3i-1}$$

   (b) 3, 5, 11, 29, 83, …,

   Solution:
   $$3^{i-1} + 2$$

   (c) 2, -3, 6, -13, 28, -59, …,

   Solution:
   $$(2^i - i + 1)(-1)^{i+1}$$

5. (10 points) Give an explicit mapping to show that the postive, even integers ($\mathbb{Z}^{\text{even},+}$) have the same cardinality as the integers divisible by 3 ($\{n$ such that $3|n\}$). You should produce a function that is a bijection. It should be an explicit formula not involving cases.

Solution:

$$f(n) = (-1)^{n/2} 3 \lfloor n/4 \rfloor$$

6. (10 points) Find all real numbers x satisfying

$$\lfloor x+1 \rfloor = \lfloor 4x \rfloor \ .$$

Show your work. HINT: Every real number x can be written as $n+\epsilon$ where n is an integer and ϵ is a real number such that $0 \leq \epsilon < 1$.

Solution: Let $x = n+\epsilon$ where n is an integer and ϵ is a real number such that $0 \leq \epsilon < 1$. Then

$$\begin{array}{ll} & \lfloor x+1 \rfloor \ = \ \lfloor 4x \rfloor \\ \text{or} & \lfloor (n+\epsilon)+1 \rfloor \ = \ \lfloor 4(n+\epsilon) \rfloor \quad \text{by substitution} \\ \text{or} & n+1+\lfloor \epsilon \rfloor \ = \ 4n+\lfloor 4\epsilon \rfloor \quad \text{taking an integer out of floor} \\ \text{or} & 3n \ = \ 1+\lfloor \epsilon \rfloor - \lfloor 4\epsilon \rfloor \quad \text{by algebra} \\ \text{or} & 3n \ = \ 1-\lfloor 4\epsilon \rfloor \quad \text{since } \lfloor \epsilon \rfloor = 0 \end{array}$$

There are four cases: If $\epsilon < 1/4$ then $3n = 1-0$, which implies $n = 1/3$, which is impossible. If $1/4 \leq \epsilon < 1/2$ then $3n = 1-1$, which implies $n = 0$. If $1/2 \leq \epsilon < 3/4$ then $3n = 1-2$, which implies $n = -1/3$, which is impossible. If $3/4 \leq \epsilon < 1$ then $3n = 1-3$, which implies $n = -2/3$, which is impossible. Thus x satisfies $\frac{1}{4} \leq x < \frac{1}{2}$.