Name & UID: ______________________________

Circle Your Section!			
0101 (10am: 3120, Ladan)	0102 (11am: 3120, Ladan)	0103 (Noon: 3120, Peter)	
0201 (2pm: 3120, Yi)	0202 (10am: 1121, Vikas)	0203 (11am: 1121, Vikas)	0204 (9am: 2117, Karthik)
0301 (9am: 3120, Huijing)	0302 (8am: 3120, Huijing)	0303 (1pm: 3120, Yi)	250H (10am: 2117, Peter)

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

1. (20 points) Let $P(n)$ be a predicate. Assume you have proven that $P(9)$ holds and that $P(n-2) \to P(n+3)$. What do you know? State your answer simply and clearly using modular arithmetic. No justification needed.

Solution: $P(n)$ holds for all $n \geq 9$ such that $n \equiv 4 \pmod 5$.

2. (20 points) We will use Mathematical Induction to prove that for $n \geq 2$

$$\sum_{j=2}^{n} \frac{1}{(j-1)j} \;=\; 1 - \frac{1}{n}$$

(a) Prove the base case.

Solution: Let $n = 2$. The summation gives

$$\sum_{j=2}^{2} \frac{1}{(j-1)j} \;=\; \frac{1}{(2-1)2} = \frac{1}{2}\,.$$

The formula gives

$$1 - \frac{1}{2} = \frac{1}{2}\,.$$

The two values are the same.

(b) What is the Inductive Hypothesis?

Solution: Assume true for $n-1$. Then

$$\sum_{j=2}^{n-1} \frac{1}{(j-1)j} \;=\; 1 - \frac{1}{n-1}\,.$$

(c) Show the Inductive Step.

Solution:

$$\begin{aligned}
\sum_{j=2}^{n} \frac{1}{(j-1)j} &= \left(\sum_{j=2}^{n-1} \frac{1}{(j-1)j}\right) + \left(\frac{1}{(n-1)n}\right) \\
&= \left(1 - \frac{1}{n-1}\right) + \left(\frac{1}{(n-1)n}\right) && \text{by IH} \\
&= 1 - \frac{n}{n(n-1)} + \frac{1}{(n-1)n} \\
&= 1 - \frac{n-1}{n(n-1)} \\
&= 1 - \frac{1}{n}\,.
\end{aligned}$$

3. (20 points) We will use Mathematical Induction to prove that for $n \geq 1$

$$\sum_{i=1}^{n} i(i!) \;=\; (n+1)! - 1$$

(a) Prove the base case.

> **Solution:** Let $n = 1$. The summation gives
>
> $$\sum_{i=1}^{1} i(i!) \;=\; 1.$$
>
> The formula gives
>
> $$(1+1)! - 1 \;=\; 1.$$
>
> The two values are the same.

(b) What is the Inductive Hypothesis?

> **Solution:** Assume true for $n - 1$. Then
>
> $$\sum_{i=1}^{n-1} i(i!) \;=\; ((n-1)+1)! - 1\ .$$

(c) Show the Inductive Step.

> **Solution:**
>
> $$\begin{aligned} \sum_{i=1}^{n} i(i!) &= \left(\sum_{i=1}^{n-1} i(i!)\right) + (n(n!)) \\ &= (((n-1)+1)! - 1) \;+\; n(n!) && \text{by IH} \\ &= (n! - 1) \;+\; n(n!) \\ &= n(n!) \;+\; n! \;-\; 1 \\ &= (n+1)(n!) \;-\; 1 \\ &= (n+1)! \;-\; 1 \end{aligned}$$

4. (20 points) We will use Mathematical Induction to show that, for all integers $n \geq 1$, $8|(3^{2n}-1)$.

   (a) Prove the base case.

   Solution: Let $n = 1$. Then

   $$3^{2n} - 1 \;=\; 3^{2\cdot 1} - 1 \;=\; 3^2 - 1 \;=\; 9 - 1 \;=\; 8.$$

   Certainly, $8|8$.

   (b) What is the Inductive Hypothesis?

   Solution: Assume true for $n-1$. Then

   $$8|(3^{2(n-1)} - 1)\ .$$

   (c) Show the Inductive Step.

   Solution:

   $$\begin{array}{lll} & 8|(3^{2(n-1)} - 1 & \text{by IH} \\ \Longrightarrow & 3^{2(n-1)} - 1 \equiv 0 \pmod 8 & \text{by the definition of divides} \\ \Longrightarrow & 3^{2n-2)} \equiv 1 \pmod 8 & \\ \Longrightarrow & 3^{2n}3^{-2} \equiv 1 \pmod 8 & \\ \Longrightarrow & 3^{2n} \equiv 3^2 \pmod 8 & \\ \Longrightarrow & 3^{2n} \equiv 9 \pmod 8 & \\ \Longrightarrow & 3^{2n} \equiv 1 \pmod 8 & \\ \Longrightarrow & 3^{2n} - 1 \equiv 0 \pmod 8 & \\ \Longrightarrow & 8|(3^{2n} - 1 & \text{by the definition of divides} \end{array}$$

5. (20 points) We will use Mathematical Induction to show that, for all integers $n \geq 2$, if $x_1, x_2, ..., x_n$ are real numbers strictly between 0 and 1 (i.e., $0 < x_i < 1$), then

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) > 1 - x_1 - x_2 - \cdots - x_n$$

(a) Prove the base case.

Solution: Let $n = 2$. The left side gives

$$(1 - x_1)(1 - x_2) \; = \; 1 - x_1 - x_2 + x_1 x_2$$

The right side gives

$$1 - x_1 - x_2$$

Clearly the left side is greater than the right side (since it has an extra term $x_1 x_2$ and $x_1, x_2 > 0$).

(b) What is the Inductive Hypothesis?

Solution: Assume true for $n - 1$. Then

$$(1 - x_1)(1 - x_2) \cdots (1 - x_{n-1}) > 1 - x_1 - x_2 - \cdots - x_{n-1}$$

(c) Show the Inductive Step.

Solution:

$$\begin{aligned}
(1 - x_1)(1 - x_2) \cdots (1 - x_n) &= (1 - x_1)(1 - x_2) \cdots (1 - x_{n-1})(1 - x_n) \\
&> (1 - x_1 - x_2 - \cdots - x_{n-1})(1 - x_n) \qquad \text{by IH} \\
&= (1 - x_1 - x_2 - \cdots - x_{n-1}) - x_n(1 - x_1 - x_2 - \cdots - x_{n-1}) \\
&= (1 - x_1 - x_2 - \cdots - x_{n-1} - x_n) + x_n(x_1 + x_2 + \cdots + x_{n-1}) \\
&> (1 - x_1 - x_2 - \cdots - x_{n-1} - x_n) + x_n \; .
\end{aligned}$$