CMSC250 Homework 9 Due: Wednesday, December 3, 2014

Name & UID: ______________________________

Circle Your Section!			
0101 (10am: 3120, Ladan)	0102 (11am: 3120, Ladan)	0103 (Noon: 3120, Peter)	
0201 (2pm: 3120, Yi)	0202 (10am: 1121, Vikas)	0203 (11am: 1121, Vikas)	0204 (9am: 2117, Karthik)
0301 (9am: 3120, Huijing)	0302 (8am: 3120, Huijing)	0303 (1pm: 3120, Yi)	250H (10am: 2117, Peter)

Question:	1	2	3	4	5	6	Total
Points:	15	15	15	15	15	25	100
Score:							

1. (15 points) Consider the following sequence:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 + 2a_0 = 3 \\
a_2 &= 1 + 2a_0 + 2a_1 = 9 \\
a_3 &= 1 + 2a_0 + 2a_1 + 2a_2 = 27 \\
&\vdots \\
a_n &= 1 + \sum_{i=0}^{n-1} 2a_i
\end{aligned}$$

We will use *Strong Induction* to prove that $a_n = 3^n$ for all $n \geq 0$.

(a) What is the Base Case:

Solution: $n = 0$
Left side $= 1$
Right side $= 3^0 = 1$

(b) What is the Inductive Hypothesis?

Solution: Assume that for all $0 \leq k < n$, $a_k = 3^k$.

(c) What is the Inductive Step?

Solution:

$$\begin{array}{lcll}
a_n & = & 1 + \sum_{i=0}^{n-1} 2a_i & \text{by definition} \\
 & = & 1 + 2\sum_{i=0}^{n-1} a_i & \text{by algebra} \\
 & = & 1 + 2\sum_{i=0}^{n-1} 3^i & \text{by IH} \\
 & = & 1 + 2\frac{3^n - 1}{3 - 1} & \text{sum of geometric series} \\
 & = & 3^n & \text{by algebra}
\end{array}$$

2. (15 points) We will use Strong Induction to show that every positive integer n can be written as the sum of *distinct* powers of 2 (i.e., the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, etc.).

   (a) What is the Base Case.

   Solution: For $n = 1$: $1 = 2^0$.

   (b) What is the Inductive Hypothesis.

   Solution: Assume that for all $1 \leq m < n$, m can be written as the sum of distinct powers of 2.

   (c) What is the Inductive Step. [Hint: Consider the cases of when n is even and odd separately.]

   Solution: If n is even then by the IH $n/2$ can be written as the sum of distinct powers of 2. Multiplying each of these powers by 2 gives n as the sum of distinct powers of 2. If n is odd then by the IH $(n-1)/2$ can be written as the sum of distinct powers of 2. Multiplying each of these powers by 2 and adding $1 = n^0$ gives n as the sum of distinct powers of 2.

3. (15 points) Show that every positive integer n can be written as the sum of distinct powers of 2 in a *unique* way. [Hint: Do not use Mathematical Induction.]

Solution: Proof by Contradiction. Assume n can be written as the sum of distinct powers of 2 in two different ways. Find the smallest powers of 2 where the two ways disagree, say 2^a and 2^b. Assume, WLOG than $a < b$. Subtract all of powers of 2 that are smaller than 2^a from both ways, producing a (possibly smaller) number that is the sum of distinct powers of 2 in two different ways. Divide both numbers by 2^a producing a (possibly smaller) number that is the sum of distinct powers of 2 in two different ways. The way that had 2^a as a power now has $2^0 = 1$ as a power and is therefore odd, but the other way is even. Thus they cannot be equal, which is a contradiction.

4. (15 points) Assume that you guess that a formula for

$$\sum_{k=1}^{n} k2^k$$

has the form $an2^n + b2^n + c$, where $n \geq 1$. We will use Constructive Mathematical Induction to derive a formula for the sum.

(a) What do you learn from the Base Case.

Solution: For $n = 1$

$$\sum_{k=1}^{n} k2^k \;=\; \sum_{k=1}^{1} k2^k \;=\; 1 \cdot 2^1 \;=\; 2\ .$$

$$an2^n + b2^n + c \;=\; a \cdot 1 \cdot 2^1 + b \cdot 2^1 + c \;=\; 2a + 2b + c\ .$$

So

$$2a + 2b + c \;=\; 2\ .$$

(b) What is the Inductive Hypothesis?

Solution: Assume for $n - 1$

$$\sum_{k=1}^{n-1} k2^k \;=\; a(n-1)2^{n-1} + b2^{n-1} + c\ .$$

(c) Show the Inductive Step.

Solution:

$$\begin{aligned} \textstyle\sum_{k=1}^{n} k2^k &= \textstyle\sum_{k=1}^{n-1} k2^k \;+\; n2^n \\ &= a(n-1)2^{n-1} + b2^{n-1} + c \;+\; n2^n && \text{by the IH} \\ &= \tfrac{a}{2}(n-1)2^n + \tfrac{b}{2}2^n + c \;+\; n2^n \\ &= (\tfrac{a}{2}+1)n2^n + (\tfrac{b-a}{2})2^n + c \\ &= an2^n + b2^n + c && \text{to make the Induction work} \end{aligned}$$

(d) Derive the constants.

Solution: The first two terms above show that $a = a/2 + 1$ and $b = (b - a)/2$.

The first equation above implies $a = 2$, then the second equation above implies $b = -2$. Using these two values with the base case implies $c = 2$. The final result is then

$$\sum_{k=1}^{n} k2^k \;=\; 2n2^n - 2 \cdot 2^n + 2 \;=\; 2(n2^n - \cdot 2^n + 1)\ .$$

5. (15 points) Consider the recurrence

$$r_n = 4r_{n-1} + r_{n-2}$$

where $r_1 = 1$ and $r_2 = 3$. We will use Constructive Mathematical Induction to derive an upper bound for r_n. Assume that $r_n \leq ab^n$. We would primarily like to upper bound b as tightly as possible, and secondarily upper bound a as tightly as possible.

(a) What do we learn from the base cases?

Solution:

$$\begin{aligned} 1 = r_1 &\leq ab^1 = ab \\ 3 = r_2 &\leq ab^2 \end{aligned}$$

(b) What is the Inductive Hypothesis?

Solution: Assume that for $k < n$, $r_k \leq ab^k$.

(c) Show the Inductive Step.

Solution:

$$\begin{aligned} r_n &= 4r_{n-1} + r_{n-2} \\ &\leq 4ab^{n-1} + ab^{n-2} \qquad \text{by IH} \\ &\leq ab^n \qquad \text{to make the Induction work} \end{aligned}$$

Dividing by ab^{n-2} and rearranging gives

$$b^2 - 4b - 1 \geq 0$$

(d) Derive the constants.

Solution: Solving for b gives

$$b \geq 2 + \sqrt{5} \ .$$

Substituting into the first base case (which dominates the second base case) gives

$$a \geq \frac{1}{2 + \sqrt{5}} = \frac{1}{b} \ .$$

Thus,

$$r_n \leq (2 + \sqrt{5})^{n-1} \ .$$

6. (25 points) A *full ternary tree* is a tree in which every node has exactly one or three children.
   (a) Guess a equation relating the number of internal nodes and leaves in a full ternary tree.

   Solution:
   $$L = 2I + 1$$

   (b) Give a Non-Structural Mathematical Induction proof of the equation.
      i. What is the Base Case:

      Solution: One node: $L = 1$ and $I = 0$. Then $2I + 1 = 2 \cdot 0 + 1 = 1 = L$.

      ii. What is the Inductive Hypothesis?

      Solution: Assume that for all trees with fewer than L leaves:
      $$L' = 2I' + 1$$

      iii. What is the Inductive Step?

      Solution: Let T be a tree with L leaves. Find an internal node for which all of its children are leaves. Remove the three children. The new tree has $L - 2$ leaves (having lost three but gained one) and $I - 1$ internal nodes (having lost one). By the IH, $L - 2 = 2(I - 1) + 1$, which implies $L = 2I + 1$.

(c) Give a recursive definition of a full trinary tree.

Solution: A *full trinary tree* is either a single node, called the root; or a single node, called the root, with three children, each of which is the root of a full trinary tree.

(d) Give a Structural Induction proof of the equation, using the definition from Part (c).

i. What is the Base Case:

Solution: One node: $L = 1$ and $I = 0$. Then $2I + 1 = 2 \cdot 0 + 1 = 1 = L$.

ii. What is the Inductive Hypothesis?

Solution: Assume that for all trees with fewer than L leaves:

$$L' = 2I' + 1$$

iii. What is the Inductive Step?

Solution: Consider a full trinary tree with more than one node. By the recursive definition it consists of a root with three children each of which is a full trinary tree. Call them T_A, T_B, and T_C. Then $L = L_A + L_B + L_C$, and $I = I_A + I_B + I_C + 1$. By the IH, $L_A = 2I_A + 1$, $L_B = 2I_B + 1$, and $L_C = 2I_C + 1$. Summing these last three equations gives:

$$\begin{aligned} L_A + L_B + L_C &= (2I_A + 1) + (2I_B + 1) + (2I_C + 1) \\ &= 2(I_A + I_B 2I_C) + 3 \end{aligned}$$

By substitution

$$L = 2(I - 1) + 3 = 2I + 1 \ .$$