CMSC250 Homework 9 Due: Wednesday, December 3, 2014

Name & UID: ____________________

Circle Your Section!			
0101 (10am: 3120, Ladan)	0102 (11am: 3120, Ladan)	0103 (Noon: 3120, Peter)	
0201 (2pm: 3120, Yi)	0202 (10am: 1121, Vikas)	0203 (11am: 1121, Vikas)	0204 (9am: 2117, Karthik)
0301 (9am: 3120, Huijing)	0302 (8am: 3120, Huijing)	0303 (1pm: 3120, Yi)	250H (10am: 2117, Peter)

Question:	1	2	3	4	5	6	Total
Points:	15	15	15	15	15	25	100
Score:							

1. (15 points) Consider the following sequence:

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 + 2a_0 = 3 \\
a_2 &= 1 + 2a_0 + 2a_1 = 9 \\
a_3 &= 1 + 2a_0 + 2a_1 + 2a_2 = 27 \\
&\vdots \\
a_n &= 1 + \sum_{i=0}^{n-1} 2a_i
\end{aligned}
$$

We will use *Strong Induction* to prove that $a_n = 3^n$ for all $n \geq 0$.

(a) What is the Base Case:

(b) What is the Inductive Hypothesis?

(c) What is the Inductive Step?

2. (15 points) We will use Strong Induction to show that every positive integer n can be written as the sum of *distinct* powers of 2 (i.e., the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, etc.).

   (a) What is the Base Case.

   (b) What is the Inductive Hypothesis.

   (c) What is the Inductive Step. [Hint: Consider the cases of when n is even and odd separately.]

3. (15 points) Show that every positive integer n can be written as the sum of distinct powers of 2 in a *unique* way. [Hint: Do not use Mathematical Induction.]

4. (15 points) Assume that you guess that a formula for

$$\sum_{k=1}^{n} k2^k$$

has the form $an2^n + b2^n + c$, where $n \geq 1$. We will use Constructive Mathematical Induction to derive a formula for the sum.

(a) What do you learn from the Base Case.

(b) What is the Inductive Hypothesis?

(c) Show the Inductive Step.

(d) Derive the constants.

5. (15 points) Consider the recurrence

$$r_n = 4r_{n-1} + r_{n-2}$$

where $r_1 = 1$ and $r_2 = 3$. We will use Constructive Mathematical Induction to derive an upper bound for r_n. Assume that $r_n \le ab^n$. We would primarily like to upper bound b as tightly as possible, and secondarily upper bound a as tightly as possible.

(a) What do we learn from the base cases?

(b) What is the Inductive Hypothesis?

(c) Show the Inductive Step.

(d) Derive the constants.

6. (25 points) A *full ternary tree* is a tree in which every node has exactly zero or three children.
   (a) Guess a equation relating the number of internal nodes and leaves in a full ternary tree.

   (b) Give a Non-Structural Mathematical Induction proof of the equation.
      i. What is the Base Case:

      ii. What is the Inductive Hypothesis?

      iii. What is the Inductive Step?

(c) Give a recursive definition of a full trinary tree.

(d) Give a Structural Induction proof of the equation, using the definition from Part (c).

i. What is the Base Case:

ii. What is the Inductive Hypothesis?

iii. What is the Inductive Step?