Name & UID: ______________________________

| Circle Your Section! | | | |
|---|---|---|---|
| 0101 (10am: 3120, Ladan) | 0102 (11am: 3120, Ladan) | 0103 (Noon: 3120, Peter) | |
| 0201 (2pm: 3120, Yi) | 0202 (10am: 1121, Vikas) | 0203 (11am: 1121, Vikas) | 0204 (9am: 2117, Karthik) |
| 0301 (9am: 3120, Huijing) | 0302 (8am: 3120, Huijing) | 0303 (1pm: 3120, Yi) | 250H (10am: 2117, Peter) |

| Question: | 1 | 2 | 3 | Total |
|---|---|---|---|---|
| Points: | 20 | 40 | 40 | 100 |
| Score: | | | | |

1. (20 points) Let $P = \{2, 3, 5, 7, 11, 13, 17, 19\}$ be the set of primes less than 20.

   (a) Use the Pigeon Hole Principle to show that there exist at least four different subsets of $P$ that all sum to the same value.

   **Solution:** There are $2^8 = 256$ subsets. The smallest possible sum is 0 (from the empty subset), and the largest possible sum 77 (the total sum). So there are at most 78 possible sums. By the PHP there are at least $\lceil 256/78 \rceil = 4$ subsets with the same value.

   (b) Show how to refine your proof to show that there exist at least five different subsets of $P$ that all sum to the same value.

   **Solution:** Remove the subsets with very small sums from the previous solution. There are eleven subsets with sum less than 11: $\{\}$, $\{2\}$, $\{3\}$, $\{5\}$, $\{7\}$, $\{2, 3\}$, $\{2, 5\}$, $\{2, 7\}$, $\{3, 5\}$, $\{3, 7\}$, $\{2, 3, 5\}$. Remove the subsets with very large sums from the previous solution. Symmetrically, there are eleven subsets with sum greater than 66. We have removed 22 subsets leaving 234 subsets. The smallest possible sum is now 11 and the largest possible sum is now 66. So there are at most $66 - 11 + 1 = 56$ possible sums. By the PHP there are at least $\lceil 234/56 = 5 \rceil$ subsets with the same value.

2. (40 points) A group of students contains five men and six women.

   (a) How many seven-player soccer teams can you form with three men and four women?

   **Solution:**
   $$\binom{5}{3}\binom{6}{4} = 10 \cdot 15 = 150 .$$

   (b) How many different lines can be formed using any two of the men and any three of the women?

   **Solution:** There are
   $$\binom{5}{2}\binom{6}{3} = 10 \cdot 20 = 200$$
   ways of picking the students, and $5! = 120$ ways to order them:
   $$\binom{5}{2}\binom{6}{3}5!$$

   (c) How many different lines can be formed using any two of the men and any three of the women if all of the women must be next to each other?

   **Solution:** First pick the five students:
   $$\binom{5}{2}\binom{6}{3}$$
   Treating the three women as one person temporarily, there 3! ways to order the "three" people. The three women themselves have 3! possible orderings. The total is
   $$\binom{5}{2}\binom{6}{3}(3!)^2$$

   (d) The six women pair up into teams of two. How many pairings are possible?

   **Solution:** The first woman can pick any of five as a teammate, the next woman can pick any of the remaining three as a teammate, and the last two are stuck with each other. This is $5 \cdot 3 = 15$.

(e) Hats are given out to some (or none or all) of the men. How many ways are there to do this?

**Solution:**

$$2^5$$

(f) I leave some number of hats in a bin and the women take all of them. How many hats do I need to leave in the bin before I know that some woman has at least 4?

**Solution:** By the generalized pigeon hole principle:

$$19$$

(g) I go to the store to buy hats for the eleven students. Hats come in four colors: red, blue, green, and yellow. How many different ways can I buy hats?

**Solution:**

$$\binom{11+4-1}{4-1} = \binom{14}{3}$$

(h) How many different ways can I buy the hats, if I have to buy at least one hat of each color?

**Solution:** This is equivalent to buying seven hats, since four hats are accounted for:

$$\binom{7+4-1}{4-1} = \binom{10}{3}$$

3. (40 points) Suppose you have a 54 card deck the contains the 52 usual cards and two jokers. The jokers are wild cards - they can represent any card of the players choosing.

   (a) How many five-card hands form four-of-a-kind without using a joker. Note that such a hand may not contain a joker (since the hand would be considered five-of-a-kind).

   **Solution:** Pick a rank of which there are 13. Then the remaining card can be any of the other 48 non-jokers. So it is

   $$13 \cdot 48$$

   (b) How many five-card hands form four-of-a-kind using exactly 1 joker. Note that such a hand may not contain two jokers (since the hand would be considered five-of-a-kind).

   **Solution:** Pick a rank of which there are 13. Then we must pick three of the four cards of that rank. Then one of the two jokers. And finally the remaining card can be any of the other 48 non-jokers. So it is

   $$13 \cdot \binom{4}{3} \cdot 2 \cdot 48 \;=\; 13 \cdot 4 \cdot 2 \cdot 48$$

   (c) How many five-card hands form four-of-a-kind using both jokers?

   **Solution:** Pick a rank of which there are 13. Then we must pick two of the four cards of that rank. Then both jokers. And finally the remaining card can be any of the other 48 non-jokers. So it is

   $$13 \cdot \binom{4}{2} \cdot 1 \cdot 48 \;=\; 13 \cdot 6 \cdot 48$$

   (d) How many five-card hands do *not* form four-of-a-kind?

   **Solution:**

   $$\binom{54}{5} - (13 \cdot 48 + 13 \cdot 4 \cdot 2 \cdot 48 + 13 \cdot 6 \cdot 48)$$