CMSC250 Section 030x: Midterm 1 Answer Key

1. (15 points) Translate each of the following English sentences into formal statements using the logical operators ($\forall, \exists, \vee, \wedge, \neg$, and $\rightarrow$). You may also use mathematical grouping and set notations, as needed. Under each translation that you provide, you will then write its negation as a *formal* statement (using the logical operators, domains and predicates).

   (a) English statement: "There are at least 2 people in this class who have blue eyes." Assume the domain $P = \{\text{all people in this class}\}$, and the predicate $B(x) =$ "x has blue eyes".

   Solution:
   Restatement: $\exists x, y \in P[B(x) \wedge B(y) \wedge (x \neq y)]$
   Negation: $\forall x, y \in P[\neg B(x) \vee \neg B(y) \vee (x = y)]$

   (b) English statement: "There is at most one positive integer that when squared equals itself." Domain: $\mathbb{Z}^+$ (all the positive integers); Predicate: $I(x) =$ "$x^2 = x$".

   Solution:
   Restatement: $\forall m, n[(m = n) \vee (m \neq m^2) \vee (n \neq n^2)]$
   Negation: $\exists m, n[(m \neq n) \wedge (m = m^2) \wedge (n = n^2)]$.

   (c) English statement: "For every rational there is an integer that when multiplied by that rational results in an integer product. Domains: $\mathbb{Z}$ and $\mathbb{Q}$ (the integers and the rationals); Predicate: $I(a, b) =$ "$a \cdot b \in \mathbb{Z}$".

   Solution:
   Restatement: $\forall q \in \mathbb{Q}\ \exists n \in \mathbb{Z}[q \cdot n \in \mathbb{Z}]$.
   Negation: $\exists q \in \mathbb{Q}\ \forall n \in \mathbb{Z}[q \cdot n \notin \mathbb{Z}]$.

2. (20 points) Logic and Circuits.

(a) Derive a DNF (Disjunctive Normal Form) representation for the output S (use the method that was presented in the textbook and was used in class).

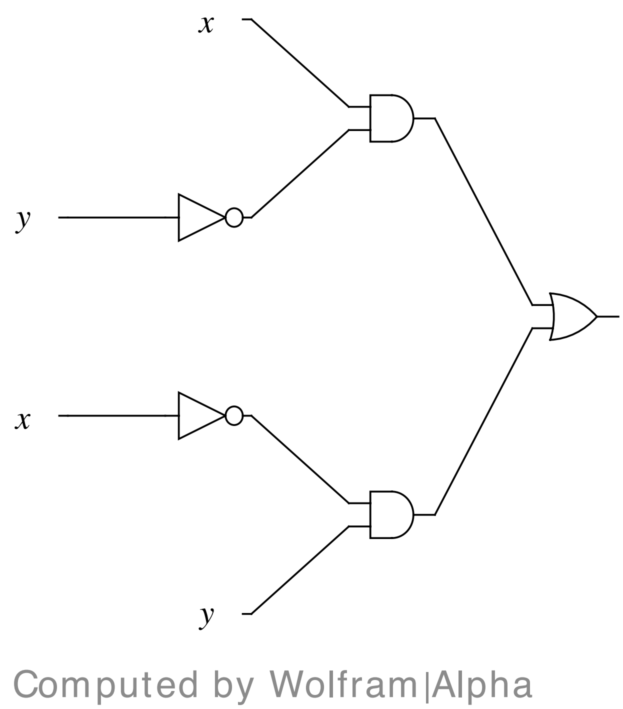
P	Q	R	S
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

Answer ____

Solution: $S = (P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

(b) Using only AND, OR, and NOT gates that take up to two inputs, draw a circuit corresponding to the expressions below for inputs x and y:

$$v = (x \wedge \neg y) \vee (\neg x \wedge y)$$



Solution:

3. (25 points) For this question, you will perform the following operation $57 - 101$ in binary arithmetic using the twos-complement arithmetic that was discussed in class. To simplify matters, assume that all numbers are 8-bits in length, and perform your calculations in parts:

   (a) Write 57 (base-10) in base-2:

   (a) 00111001

   (b) Write 101 (base-10) in base-2:

   (b) 01100101

   (c) Convert the base-2 number from part(b) into its twos-complement form:

   (c) 10011011

   (d) Perform the operation by summing the last two numbers, i.e., the numbers from part(a) and part(c); leave your answer in twos-complement form:

   (d) 11010100

   (e) Using the twos-complement, rewrite the number you computed in part(d) as a positive base-2 integer:

   (e) 00101100

4. (20 points) For this (two part) problem, assume that $\times$, $=$, $\neq$, 0, and 1 have their usual meanings. All domains considered contain at least 0 and 1. For the following sentence

$$(\forall x)[x \neq 0 \rightarrow (\exists y)[x \times y = 1]],$$

(a) Give (specify) a non-empty domain where the sentence is true.

> **Solution:** $\mathbb{Q}$: note that for every $x = a/b$ there is a $y = b/a$, such that $xy = 1$.

(b) Give (specify) a non-empty domain where the sentence is false.

> **Solution:** $\mathbb{Z}$: note that an infinite number of integers contain no multiplicative inverses in the integers, e.g., let $x = 2$, now no integer y exists such that $xy = 1$.

5. (20 points) Prove that for integers a and b, if $a^2 + b^2$ is odd then $a + b$ is odd. You may use the fact that the sum of evens is even, the sum of two odds is even and the sum of an even and an odd is odd.

   Your proof should employ contraposition and cases.

Solution:

Proof. By contraposition, we need to show that $a + b$ is even implies that $a^2 + b^2$ is even. We consider the following cases:

Case. a and b even. Then let $a = 2k$ and $b = 2l$, for integers k and l, and by substitution

$$a^2 + b^2 = (2k)^2 + (2l)^2 = 4k^2 + 4l^2 = 2(2k^2 + 2l^2).$$

Because the integers are closed under addition, this last term is even.

Case. a and b odd. Then let $a = 2k + 1$ and $b = 2l + 1$, for integers k and l. Substituting,

$$\begin{aligned} a^2 + b^2 &= (2k+1)^2 + (2l+1)^2 = (4k^2 + 4k + 1) + (4l^2 + 4l + 1) \\ &= 4k^2 + 4l^2 + 4k + 4l + 2 = 2(2k^2 + 2l^2 + 2l + 2k + 1), \end{aligned}$$

and that last term is an even number.

Case. a is even and b is odd. The sum of an even and an odd integer is odd, and therefore the implication is vacuously true, and there is nothing left to prove.

Having shown that in each case the implication holds, we conclude that the proposition is true. $\square$

6. (20 points) Prove that for any rational numbers a and b, $a \neq 0$ that $a\sqrt{2} - b$ is irrational.

Solution:

Proof. BWOC assume not. Then $a\sqrt{2} - 2$ is rational, meaning that $a\sqrt{2} - b = r$, where $r \in \mathbb{Q}$. Rearranging terms gives:

$$a\sqrt{2} - b = r = a\sqrt{2} = r + b = \sqrt{2} = \frac{r+b}{a}, \text{ where } a \neq 0.$$

But this means that an irrational number, $\sqrt{2}$, can be expressed as a rational number, $\frac{r+b}{a}$, which is a contradiction. $\square$