CMSC250: Midterm # 2—Answer Key

Fall 2104

CMSC250, Fall 2014: Midterm 2—Key

1. (15 points) Using the set properties and the rules of logic (provided as an appendix to this document), prove that for sets A and B: $A-(A-B)=A\cap B$. Your proof must reference the set-properties or a rules of logic at each step.

Solution:

$$
\begin{aligned}
A\cap(A-B) &= A\cap\overline{(A\cap\bar{B})} && \text{Definition of set difference}\\
&= A\cap(\bar{A}\cup B) && \text{DeMorgans laws and negation}\\
&= (A\cap\bar{A})\cup(A\cap B) && \text{Distributive property}\\
&= \emptyset\cup(A\cap B) && \text{Intersection with Complement}\\
&= A\cap B && \text{Union with } \emptyset
\end{aligned}
$$

2. (20 points) Let $X = \mathbb{R} - \{-1\}$, and define $f\colon X \longrightarrow \mathbb{R}$ by the rule: $f(x) = \dfrac{2x}{x+1}$.
Prove or disprove the following assertions about f:

(a) f is injective (one-to-one).

Solution: To show that f is injective, choose arbitrary x_1 and x_2 in X and assume that $f(x_1) = f(x_2)$. Then, by the definition of f

$$\frac{2x_1}{x_1+1} = \frac{2x_2}{x_2+1} = 2x_1(x_2+1) = 2x_2(x_1+1).$$

This gives

$$2x_1x_2 + 2x_1 = 2x_1x_2 + 2x_2.$$

Thus, $2x_1 = 2x_2$ and $x_1 = x_2$, which needed to be shown.

(b) f is surjective (onto).

Solution: f is *not* surjective. To see this, we show that for all $x \in \mathbb{R}$, $f(x) \neq 2$. Assume that $x \in X$ and $f(x) = 2$. Then by the definition of f we have

$$\frac{2x}{x+1} = 2 \text{ which gives } 2x = 2x + 2,$$

but this is not possible, and therefore 2 remains unmapped and f is *not* onto. Alternatively, assume that f is surjective, then it has a right inverse. Setting

$$y = \frac{2x}{x+1} \text{ gives an inverse } x = -\frac{y}{y-2} \text{ or } x = \frac{y}{2-y}.$$

Observe that this function is undefined when $y = 2$. Thus, for the point 2 no inverse exists from the codomain of f to its domain, thus f is not surjective.

3. (15 points) Use congruence modulo m to prove the following lemma:

Lemma. *If* $3 \mid n^2$ *then* $3 \mid n$ *for all integers* n.

Solution:

Proof. We show the contrapositive: if $3 \nmid n$ then $3 \nmid n^2$.

Assuming the hypothesis show that $n^2 \not\equiv 0 \bmod 3$ for any of the representatives of 3, which are $\{1, 2\}$:

$$\begin{array}{lll} n = 1 & (1)^2 = 1 & 1 \not\equiv 0 \bmod 3 \\ n = 2 & (2)^2 = 4 & 4 \not\equiv 0 \bmod 3 \end{array}$$

Having shown that the contrapositive holds, we conclude that the original proposition is true. □

4. (15 points) Use the lemma from the previous question to prove that $\sqrt{3}$ is irrational.

Solution:

Proof. BWOC assume not, then $\sqrt{3} = a/b$, where $a, b \in \mathbb{Z}$, $b \neq 0$, and a and b are relatively prime, i.e., $\gcd(a, b) = 1$.

Then squaring both sides gives

$$3 = \frac{a^2}{b^2} \text{ which gives } 3b^2 = a^2.$$

Because 3 divides a^2, by the lemma above 3 divides a. Letting $3n = a$, some integer n, and substituting this quantity gives $3b^2 = (3n)^2$ or $b^2 = 3n^2$. By simliar reasoning, we may apply the lemma from the previous problem to this equation and conclude that b is also divisible by 3. But this is a contradiction because we assumed that both a and b had a greatest common divisor of 1.

Having shown that the negation of the original implication has given a contradiction, we may conclude that the original proposition is true. □

5. (15 points) Use the Fundamental Theorem of Arithmetic (Unique Factorization Theorem) to prove that $\sqrt[3]{5}$ is irrational.

Solution:

Proof. BWOC assume not, then $\sqrt[3]{5} = a/b$, where $a, b \in \mathbb{Z}$, $b \neq 0$ and $\gcd(a, b) = 1$. Cubing both sides of the equation gives $5 = a^3/b^3$, or $5b^3 = a^3$.

By the Unique Factorization Theorem, we can write a and b as a unique product of prime numbers:

$$a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} \text{ and } b = q_1^{b_1} q_2^{b_2} \dots q_n^{b_n}.$$

Because each side is cubed, their prime factors appear as

$$a^3 = p_1^{3a_1} p_2^{3a_2} \dots p_k^{3a_k} \text{ and } b^3 = q_1^{3b_1} q_2^{3b_2} \dots q_n^{3b_n}.$$

Re-write $5b^3 = a^3$ as

$$5q_1^{3b_1} q_2^{3b_2} \dots q_n^{3b_n} = p_1^{3a_1} p_2^{3a_2} \dots p_k^{3a_k}.$$

Note that because $5b^3 = a^3$ some p in the factor set for a^3 must be a 5. But, then the corresponding prime factor q belonging to b^3 has an additional factor of 5. And, by the Unique Factorization Theorem, two numbers that are equal cannot differ by a number of times that a prime factor appears in either's factorization. This is a contradiction. $\square$

6. (15 points) Prove by induction that for all positive integers n:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

(a) Show the base case.

Solution: When $n = 1$, both sides of the equation equal $1/2$.

(b) State the Inductive Hypothesis:

Solution: Assume that the statement holds for some $k \geq 1$

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$$

in order to show that

$$\sum_{k=1}^{n+1}\frac{1}{k(k+1)}=\frac{n+1}{n+2}.$$

(c) Show the inductive step:

Solution:

$$\begin{aligned}\sum_{k=1}^{n+1}\frac{1}{k(k+1)} &= \sum_{k=1}^{n}\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}\\ &= \frac{n}{n+1}+\frac{1}{(n+1)(n+2)} \quad \text{By IH}\\ &= \frac{n(n+2)+1}{(n+1)(n+2)} \quad \text{Combining terms}\\ &= \frac{(n+1)^2}{(n+1)(n+2)}\\ &= \frac{n+1}{n+2}.\end{aligned}$$

Having shown the base case and the inductive step, we conclude by the principle of induction that the statement is true.

7. (15 points) Prove by induction that $15^n + 6$ is divisible by 7 for all positive integers n.

(a) Show the base case:

Solution: Let $n = 1$ then $15 + 6 = 21$ and $7 \mid 21$.

(b) State the Inductive Hypothesis:

Solution: Assume that $15^k + 6$ is divisible by 7 for some $k \geq 1$ in order to show that $15^{k+1} + 6$ is divisible by 7.

(c) Show the Inductive Step:

Solution: Observe that $15^{k+1} + 6 = 15(15^k + 6)$. And by the Inductive Hypothesis we can re-write this as $15^k = 7n - 6$ for some integer n. Substituting this into the inductive step gives

$$\begin{aligned} 15(7n-6)+6 &= 15(7n-90)+6 \\ &= 15(7n) - 84 \\ &= 7(15n) - 7(12) \\ &= 7(15n - 12) \end{aligned}$$

And this last expression is clearly divisible by 7, which needed to be shown.

Having shown the base case and the inductive step, we conclude by the principle of induction that the statement is true.