What is a predicate?

A "predicate" is a statement involving variables over a specified "domain" (set).

Example

Domain (set)	Predicate
Integers ($\mathbb{Z}$)	$S(x)$: x is a perfect square
Reals ($\mathbb{R}$)	$G(x, y)$: $x > y$
Computers	$A(c)$: c is under attack
Computers; People	$B(c, p)$: c is under attack by p

Quantification

- Existential quantifier: $\exists$ (exists)
- Universal quantifier: $\forall x$ (for all)

Domain D, Subdomain D' subset of D

- $(\exists x)[P(x)]$: Exists x in D such that $P(x)$ is true.
- $(\exists x \in D')[P(x)]$: Exists x in D' such that $P(x)$ is true.
- $(\forall x)[P(x)]$: For all x in D, $P(x)$ is true.
- $(\forall x \in D')[P(x)]$: For all x in D', $P(x)$ is true.

Establishing Truth and Falsity

- To show $\exists$ statement is true:
  Find an example in the domain where it is true.
- To show $\exists$ statement is false:
  Show false for every member of the domain.
- To show $\forall$ statement is true:
  Show true for every member of the domain.
- To show $\forall$ statement is false:
  Find an example in the domain where it is false.

There are other methods!!!

Negation of Quantified Statements

Example

It is not the case that there is some cat that can fly:

- Some cats cannot fly.
- Only some cats can fly.
- Cats can only fly sometimes.
- All cats cannot fly.
- Watch me throw this cat out the window.

Predicate $F(x)$: x can fly.

$$\neg(\exists x \in \text{cats})[F(x)] \quad \equiv \quad (\forall x \in \text{cats})[\neg F(x)]$$

Negation of Quantified Statements

Example

Not everybody likes me:

- Nobody likes me.
- Everybody doesn't like me.
- Somebody doesn't like me.

Predicate $L(x)$: person x likes me.

$$\neg(\forall x \in \text{people})[L(x)] \quad \equiv \quad (\exists x \in \text{people})[\neg L(x)]$$

Vacuous cases for universally quantified statements

- All prime numbers that are greater than 10 are the sum of two squares.
- All students in this class who are more than ten feet tall have green hair.

Are these statements True or False?
How do we show it?

Quantified conditional statements

- Universal conditional statement: $(\forall x \in D)[P(x) \rightarrow Q(x)]$
  - Contrapositive: $(\forall x \in D)[\neg Q(x) \rightarrow \neg P(x)]$
  - Converse: $(\forall x \in D)[Q(x) \rightarrow P(x)]$
  - Inverse: $(\forall x \in D)[\neg P(x) \rightarrow \neg Q(x)]$
- Existential conditional statement: $(\exists x \in D)[P(x) \rightarrow Q(x)]$
  - Contrapositive: $(\exists x \in D)[\neg Q(x) \rightarrow \neg P(x)]$
  - Converse: $(\exists x \in D)[Q(x) \rightarrow P(x)]$
  - Inverse: $(\exists x \in D)[\neg P(x) \rightarrow \neg Q(x)]$

Multiple Quantifiers (Same Type)

Domains: set of all chairs (C); set of all people (P).

Predicate $S(p, c)$: Person p is sitting on chair c.

- Existential
  - $(\exists p, \exists c)[S(p, c)]$ There is a person sitting on a chair.
  - $(\exists c, \exists p)[S(p, c)]$ There is a chair with someone sitting on it.
  - Alternatively: $(\exists c, p)[S(p, c)]$ or $(\exists p, c)[S(p, c)]$
- Universal
  - $(\forall p, \forall c)[S(p, c)]$ All people are sitting on all chairs.
  - $(\forall c, \forall p)[S(p, c)]$ All chairs have all people sitting on them.
  - Alternatively: $(\forall c, p)[S(p, c)]$ or $(\forall p, c)[S(p, c)]$

Multiple Quantifiers II

Example

The order of $\forall$ and $\exists$ matters!

- $(\forall p, \exists c)[S(p, c)]$ Everybody is sitting on a chair.
- $(\exists c, \forall p)[S(p, c)]$
  There is some chair that everybody is sitting on.
- $(\forall c, \exists p)[S(p, c)]$
  Every chair has somebody sitting on it.
- $(\exists p, \forall c)[S(p, c)]$
  There is some person sitting on all of the chairs.

Multiple Quantifiers III

Example

Domain: Set of integers ($\mathbb{Z}$)

- $(\forall m,\ \exists n)[n > m]$ True
  Every number has some other number larger that it.
- $(\exists n,\ \forall m)[n > m]$ False
  There exists a number larger than all other numbers.

Meanings and Negations of Multiply Quantified Statements

English: All people like some cat.

Predicate $L(p, c)$: Person p likes cat c.
Do we mean:

$$(\forall p, \exists c)[L(p, c)] \quad \text{or} \quad (\exists c, \forall p)[L(p, c)] \text{ ?}$$

Take the negation:

$$\neg(\forall p, \exists c)L(p, c) \equiv (\exists p, \forall c)\neg L(p, c)$$

$$\neg(\exists c, \forall p)L(p, c) \equiv (\forall c, \exists p)\neg L(p, c)$$

Quantified Cardinality

Example

Domains: set of all students (S); set of all colleges (C).

Predicate $A(s,c)$: Student s attends college c.

- Exactly one student attends college.

$(\exists s \in S, \exists c \in C)[A(s,c) \wedge \neg(\exists t \in S, \exists d \in C)[(t \neq s) \wedge A(t,d)]]$
$(\exists s \in S, \exists c \in C)[A(s,c) \wedge (\forall t \in S, \forall d \in C)\neg[(t \neq s) \wedge A(t,d)]]$
$(\exists s, \exists c \in C)[A(s,c) \wedge (\forall t \in S, \forall d \in C)[(t = s) \wedge \neg A(t,d)]]$
$(\exists s \in S, \exists c \in C)[A(s,c) \wedge (\forall t \in S, \forall d \in C)[A(t,d) \rightarrow (t = s)]]$

- At most one student attends college.

$$(\forall s,t \in S,\ \forall c,d \in C)[(A(s,c) \wedge A(t,d)) \rightarrow (s = t)]$$

- At least two students attend college.

$$(\exists s,t \in S,\ \exists c,d \in C)[A(s,c) \wedge A(t,d) \wedge (s \neq t)]$$

Rules of Inference

Universal Instantiation	$\dfrac{(\forall x)[P(x)]}{\therefore P(c)}$
Universal Generalization	$\dfrac{P(c) \text{ for arbitrary element } c}{\therefore (\forall x)[P(x)]}$
Existential Instantiation	$\dfrac{(\exists x)[P(x)]}{\therefore P(c) \text{ for some element } c}$
Existential Generalization	$\dfrac{P(c) \text{ for some element } c}{\therefore (\exists x)[P(x)]}$

Using Rules of Inference

Example

A student in this class has not read the book.

Everyone in this class passed the first exam.

$\therefore$ Someone who passed has not read the book.

Number	Statement	Justification
(1)	$(\exists x)[C(x) \wedge \neg B(x)]$	Premise
(2)	$(\forall x)[C(x) \wedge P(x)]$	Premise
$\vdots$	$\vdots$	$\vdots$
(?)	$(\exists x)[P(x) \wedge \neg B(x)]$	Conclusion

Do in class.

What is a proof?

A good proof should have:

- A clear statement of what is to be proved (labeled as Theorem, Lemma, Proposition, or Corollary).
- The word "Proof" to indicate where the proof starts.
- A clear indication of flow.
- A clear justification for each step.
- A clear indication of the conclusion.
- The abbreviation "QED" ("Quod Erat Demonstradum" or "that which was to be proved") or equivalent to indicate the end of the proof.

Summary of Proof Methods

- Direct proof
- Proof by contraposition
- Proof by contradiction
- Exhaustive Proof
- Proof by cases

Statement of Theorems

The following are equivalent:

- The sum of two positive integers is positive.
- If m, n are positive integers then their sum $m + n$ is a positive integer.
- For all positive integers m, n their sum $m + n$ is a positive integer.
- $(\forall m, n \in \mathbb{Z})[((m > 0) \wedge (n > 0)) \rightarrow ((m + n) > 0)]$

Number Definitions

Definition

An integer n is *even* if $n = 2k$ for some integer k, and is *odd* if $n = 2k + 1$ for some integer k.

Definition

A number q is *rational* if there exist integers a, b with $b \neq 0$ such that $q = a/b$.

Definition

A real number that is not rational is *irrational*.

Closure

- $\mathbb{Z}$ is closed under addition.

  If $a, b \in \mathbb{Z}$ then $a + b \in \mathbb{Z}$.

- $\mathbb{Q}^{\neq 0}$ is closed under division.

  If $q, r \in \mathbb{Q}^{\neq 0}$ then $\frac{q}{r} \in \mathbb{Q}^{\neq 0}$.

- $\mathbb{Z}^{\neq 0}$ is *not* closed under division.

  $\frac{3}{5} \notin \mathbb{Z}^{\neq 0}$.

Direct Proofs

- The square of an even number is even.
- The product of two odd numbers is odd.
- The sum of two rational numbers is rational.

$\boxed{\text{Do in class.}}$

Proof by Contraposition

- If $3n + 2$ is odd, where n is an integer, then n is odd.
- If n^2 is even, where n is an integer, then n is even.
- If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. What is the domain for n?

Do in class.

Proof by Contradiction

- At least four of any 22 days must fall on the same day of the week.
- $\sqrt{2}$ is irrational.

$\boxed{\text{Do in class.}}$

Proofs of Equivalence

- If n is an integer, then n is odd if and only if n^2 is odd.
- The follow statements about the integer n are equivalent:
  - n is even.
  - $n - 1$ is odd.
  - n^2 is even.

$\boxed{\text{Do in class.}}$