Set Theory

Set definitions

Definition: A *set* is a collection of objects.

Examples: $A = \{1,2,3\}$

$B = \{x \in Z \mid -4 < x < 4\}$

$C = \{x \in Z^+ \mid -4 < x < 4\}$

A set is completely defined by its elements, i.e.,

$\{a,b\} = \{b,a\} = \{a,b,a\} = \{a,a,a,b,b,b\}$

More set concepts

- The universal set *U* is the set of all elements under consideration
- A set can be finite or can be infinite
- For a set *S*, n(*S*) or |*S*| are used to refer to the cardinality of *S*, which is the number of elements in *S*
- The symbol $\in$ means "is an element of"
- The symbol $\notin$ means "is not an element of"

Subset

- $A \subseteq B \leftrightarrow (\forall x \in U)[x \in A \rightarrow x \in B]$

  A is contained in B

  B contains A

- $A \not\subseteq B \leftrightarrow (\exists x \in U)[x \in A \wedge x \notin B]$

- Relationship between membership and subset:

  $$(\forall x \in U)[x \in A \leftrightarrow \{x\} \subseteq A]$$

- Definition of set equality: $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

Proper subset

$$A \subset B \leftrightarrow A \subseteq B \wedge A \neq B$$

Do these represent the same sets or not?

$X = \{x \in \mathbf{Z} \mid (\exists p \in \mathbf{Z})[x = 2p]\}$

$Y = \{y \in \mathbf{Z} \mid (\exists q \in \mathbf{Z})[y = 2q - 2]\}$

$\mathrm{A} = \{x \in \mathbf{Z} \mid (\exists i \in \mathbf{Z})[x = 2i + 1]\}$

$\mathrm{B} = \{x \in \mathbf{Z} \mid (\exists i \in \mathbf{Z})[x = 3i + 1]\}$

$\mathrm{C} = \{x \in \mathbf{Z} \mid (\exists i \in \mathbf{Z})[x = 4i + 1]\}$

Formal definitions of set operations

Union: $A \cup B = \{x \in A \vee x \in B\}$

Intersection: $A \cap B = \{x \in A \wedge x \in B\}$

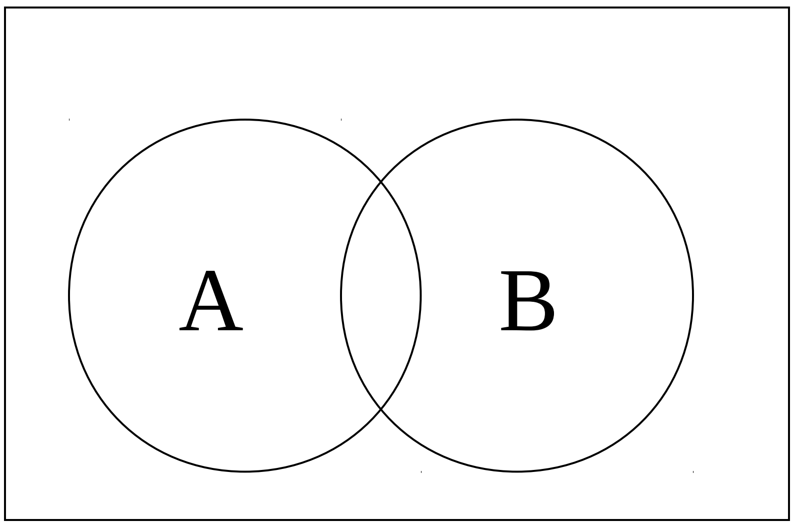
Complement: $A^c = A' = \overline{A} = \{x \in U \mid x \notin A\}$

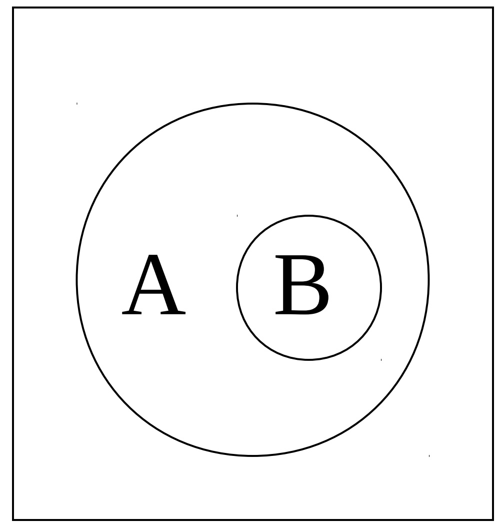
Difference: $A - B = \{x \in A \wedge x \notin B\}$

$$A - B = A \cap \overline{B}$$

Venn Diagrams

Sets are represented as regions to illustrate relationships between them.





The empty set and its properties

The empty set $\varnothing$ has no elements, so $\varnothing = \{\}$.

1. $(\forall \text{ sets } X)[\varnothing \subseteq X]$
2. There is only one empty set.
3. $(\forall \text{ sets } X)[X \cup \varnothing = X]$
4. $(\forall \text{ sets } X)[X \cap \overline{X} = \varnothing]$
5. $(\forall \text{ sets } X)[X \cap \varnothing = \varnothing]$
6. $\overline{U} = \varnothing$
7. $\overline{\varnothing} = U$

The Cartesian Product

- The Cartesian product of sets *A* and *B* is defined as

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

- $|A \times B| = |A|\,|B|$ (size of *A* times size of *B*)

Ordered n-tuples

- An ordered n-tuple takes order and multiplicity into account
- The tuple $(x_1,x_2,x_3,\ldots,x_n)$
  - has n values
  - not necessarily distinct
  - order matters (unlike sets)
- $(x_1,x_2,x_3,\ldots,x_n) = (y_1,y_2,y_3,\ldots,y_n) \leftrightarrow (\forall i \in 1 \leq i \leq n)[x_i = y_i]$
- 2-tuples are called pairs, and 3-tuples are called triples

Examples

- $(1,3,2) \neq (3,2,1)$
- $\{1,3,2\} = \{3,2,1\}$
- $(1,1,3,2)$ is a fine 4-tuple
- $(1,1,3,2) \neq (1,3,2)$
- $\{1,1,3,2\} = \{1,3,2\}$

Disjoint Sets

- A and B are disjoint

  $\leftrightarrow$ A and B have no elements in common

  $\leftrightarrow (\forall x \in U)[x \in A \rightarrow x \notin B]$

- $A \cap B = \emptyset \leftrightarrow A$ and B are disjoint sets

- If A and B are disjoint then $|A \cup B| = |A| + |B|$

  What if A and B are not disjoint?

Power set

$\mathcal{P}(A)$ = the set of **all** subsets of A

Examples: What are

$\mathcal{P}(\{a,b\})$?
$\mathcal{P}(\{a,b,c\})$?
$\mathcal{P}(\{a\})$?
$\mathcal{P}(\varnothing)$?
$\mathcal{P}(\{\varnothing\})$?

Properties of power sets

- $(\forall \text{ sets } A,B)[A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)]$

- $(\forall \text{ sets } A)[\ |\mathcal{P}(A)| = 2^{|A|}\ ]$

Proof in class

Properties of sets

- Inclusion $A \cap B \subseteq A$ $A \cap B \subseteq B$

  $A \subseteq A \cup B$ $B \subseteq A \cup B$

- Transitivity $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

More Properties of Sets__

- DeMorgan's for complement

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

- Distribution of union and intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- There are a number of others

Analogies between Sets and Logic

- $A \cup B$ is analogous to $A \vee B$
- $A \cap B$ is analogous to $A \wedge \mathrm{B}$
- $\overline{A}$ is analogous to $\neg A$
- What is $A \rightarrow$ implies B analogous to?

Using Venn diagrams to help find counterexamples

$$A \cup (B \cap C) =?= (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) =?= (A \cup B) \cap C$$

Deriving new properties using Venn diagrams

$$B - (A \cap C) = (B - A) \cup (B - C)$$

$$A - B = A - (A \cap B)$$

$$A \subseteq B \wedge A \subseteq C \rightarrow A \subseteq (B \cap C)$$

Proof in class

Partitions of a Set

- A collection of nonempty sets $\{A_1,A_2,\ldots,A_n\}$ is a partition of the set A if and only if

  1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$

  2. $A_1,A_2,\ldots,A_n$ are mutually disjoint

- An infinite set can be partitioned. The partitions can be infinite, or can be finite.

Russell's Paradox

- A set can be an element or member of itself.

Example: List of all lists.

Set of infinite sets.

- Consider the set $S = \{A \mid A \text{ is a set and } A \notin A\}$
  - Is S an element of itself?

Russell's Paradox Continued

- $S = \{A \mid A \text{ is a set and } A \notin A\}$
  - Is S an element of itself?
  - Option 1: YES! $S \in S$. Then by def of S, $S \notin S$.
  - Option 2: NO! $S \notin S$. Then by def of S, $S \in S$.
  - Upshot: S does not exist!!!

The Halting Problem

Does such a computer program exist?

INPUT: Program **P**, input **x**

OUTPUT: YES if program **P** on input **x** halts

NO if program **P** on input **x** does not halt

An approach: Run **P** on **x** and see what happens.

Does not work!: If it does not halt we will never know.

How about a different approach???

Proof for the Halting Problem

Suppose there is a program **`Halt(P,x)`** for the halting problem.

Create a new program **`Test`**:

```
procedure Test(P)
  if Halt(P,P)outputs YES then loop forever;
  if Halt(P,P)outputs NO then STOP;
end procedure
```

Now run **`Test(Test)`**:

- if **`Test(Test)`** halts, then **`Halt(Test,Test)`** outputs YES; hence **`Test(Test)`** does not halt.
- if **`Test(Test)`** does not halt, then **`Halt(Test,Test)`** outputs NO; hence **`Test(Test)`** halts.