# CMSC351 - Fall 2014, Final Exam

Type A

WAIT FOR INSTRUCTIONS BEFORE BEGINNING

HONOR PLEDGE: "I pledge on my honor that I have not given or received any unauthorized assistance on this examination."

Signature and UID:

PRINT Name:

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- Write your answers with enough detail about your approach and concepts used, so that the grader will be able to understand it easily.
- The sum of the grades is 105, but your grades would be out of 100 (thus you get 5 bonus points by solving all the problems).
- In this exam, $n$ denotes the number of vertices and $m$ denotes the number of edges.

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Select the best choice for the first 8 problems and mark it by X in the table below:

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| a | X | | | | | | | |
| b | | | X | | | X | | |
| c | | | | | | | X | |
| d | | | | X | | | | X |
| e | | X | | | X | | | |

DO NOT WRITE BELOW THIS LINE

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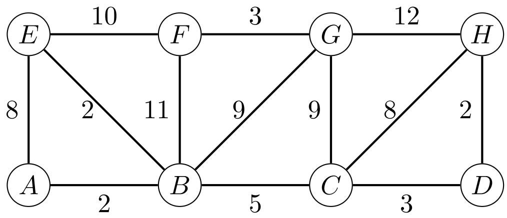
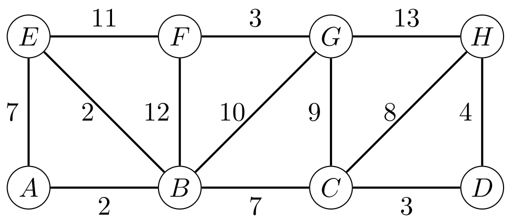
| | |
|---|---|
| Problems 1-8 | |
| Problem 9 | |
| Problem 10 | |
| Problem 11 | |
| **TOTAL** | |

**Problem 1** **(5 points)** Run Dijkstra's algorithm on the weighted graph below, using vertex $A$ as the source. Write the vertices in the order which they are marked.




a) $A, B, E, C, D, G, H, F$ b) $A, B, E, C, D, H, G, F$ c) $A, B, E, F, G, C, D, H$
d) $A, E, B, C, D, G, H, F$ e) $A, B, C, D, E, G, F, H$

**Problem 2** **(5 points)** Run Prim's algorithm on the weighted graph below, using vertex $A$ as the source. What is the sum of the weights of the first, the third, and the fifth edges that are added to the output of Prim's algorithm?




a) 9 b) 10 c) 11 d) 12 e) 13

**Problem 3** **(5 points)** Find an optimal parenthesization of $A_1 \times A_2 \times A_3 \times A_4$ where $A_1$ is a $2 \times 4$ matrix, $A_2$ is a $4 \times 5$ matrix, $A_3$ is a $5 \times 3$ matrix, and $A_4$ is a $3 \times 2$ matrix. What is the cost of an optimal solution?

a) 78 b) 82 c) 86 d) 90 e) 96

**Problem 4** **(5 points)** Consider an execution of the DFS algorithm on directed graph $G$. Which of the following statements is **correct**?

(I) Edge $(u, v)$ is a cross-edge iff $v.finish < u.start$.

(II) Edge $(u, v)$ is a back-edge iff $u.finish < v.finish$.

(III) Edge $(u, v)$ is a tree-edge iff $u.start < v.start$ and $u.finish > v.finish$.

a) I b) II c) III d) I and II e) II and III

**Problem 5** **(5 points)** The following procedure computes $2^n$ for any non-negative integer $n$. What is its running time?

**Algorithm 1** POWER($n$)

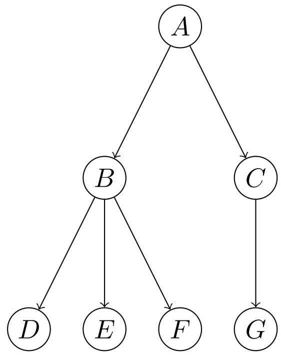
```
if n=0 then
    return 1
end if
return POWER(n − 1) + POWER(n − 1)
```

a) $O(\log n)$ b) $O(n)$ c) $O(n \log n)$ d) $O(n^2)$ e) $O(2^n)$

**Problem 6 (5 points)** For a graph with $\Theta(n)$ weighted edges, what is the best running time for Dijkstra's algorithm?

a) $O(n)$ b) $O(n \log n)$ c) $O(n \log^2 n)$ d) $O(n^2)$ e) $O(n^2 \log n)$

**Problem 7 (5 points)** Let the following tree be a BFS tree of a directed graph $G$. Which of the following edges cannot be in graph $G$?




a) $(D, A)$ b) $(B, C)$ c) $(A, E)$
d) $(B, G)$ e) $(F, C)$

**Problem 8 (5 points)** Consider the following as the output of the DFS algorithm on an undirected graph. Which of the following edges is not in the corresponding DFS tree?

| Vertex | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| Start Time | 1 | 2 | 5 | 3 | 11 | 6 | 8 |
| Finish Time | 14 | 13 | 10 | 4 | 12 | 7 | 9 |

a) $(B, C)$ b) $(B, D)$ c) $(C, G)$ d) $(B, G)$ e) $(A, B)$

**Problem 9** **(25 points)** Answer the following questions regarding Kruskal's algorithm for finding a minimum spanning tree.

(a) (5 points) Describe briefly Kruskal's algorithm?

(b) (5 points) Assume procedures $\textsc{find}(u)$ and $\textsc{union}(u, v)$ are given. Write the pseudo-code for Kruskal's algorithm.

(c) (5 points) Write pseudo-codes for both $\textsc{find}(u)$ and $\textsc{union}(u, v)$ which run in $O(\log n)$.

(d) (10 points) Prove the correctness of the algorithm.

**Solution:** Look at Chapter 23 of the book by Cormen et al.

**Problem 10** **(20 points)** Matrix $M$ with $n$ rows and $m \geq n$ columns is given. Let cell $(1, 1)$ be the lower left corner of the matrix. We want to choose exactly one cell from each row such that:

- For every $1 \leq k < n$, the selected cell from the $k$-th row is on the left of the selected cell from the $k+1$-th row.
- The sum of all selected cells is maximized.

The goal is to design a dynamic program which runs in $O(nm)$ for solving this problem. For every $i \leq n$ and $j \leq m$, consider submatrix $M'$ with the first $i$ rows and the first $j$ columns of matrix $M$, and let $A[i, j]$ be the optimum solution for submatrix $M'$.

(a) (10 points) Consider a decision regarding cell $(i, j)$. Give a formula for computing $A[i, j]$ based on the solutions of subproblems.

**Solution:** Here is the formula:

$$A[i, j] = \max\{A[i, j-1], A[i-1, j-1] + M[i, j]\}$$

(b) (3 points) What are base cases? What would be their values?

**Solution:** Here are the base cases:

$$\forall 0 \leq j \leq m, A[0, j] = 0$$

$$\forall 1 \leq i \leq n, A[i, 0] = -\infty$$

(c) (7 points) Write a pseudo-code for solving the problem?

**Solution:**

**Algorithm 2** DYNAMIC$(M, n, m)$

```
for i = 1 to n do
    A[i, 0] = −∞
end for
for j = 0 to m do
    A[0, j] = 0
end for
for i = 1 to n do
    for j = 1 to m do
        A[i, j] = max{A[i, j − 1], A[i − 1, j − 1] + M[i, j]}
    end for
end for
return A[n, m]
```

**Problem 11 (20 points)** Consider the following implementation of the Floyd-Warshall algorithm. Assume $w_{ij} = \infty$ when there is no edge between vertex $i$ and vertex $j$, and assume $w_{ii} = 0$ for every vertex $i$.

**Algorithm 3** FOLYD-WARSHALL($G$)

```
for i = 1 to n do
    for j = 1 to n do
        A[i, j, 0] = w_ij
        P[i, j] = −1
    end for
end for
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            A[i, j, k] = A[i, j, k − 1]
            if A[i, j, k] > A[i, k, k − 1] + A[k, j, k − 1] then
                A[i, j, k] = A[i, k, k − 1] + A[k, j, k − 1]
                P[i, j] = k
            end if
        end for
    end for
end for
```

(a) (12 points) Assume matrix $P$, the output of the above algorithm, is given. Design an algorithm for finding the shortest path from $u$ to $v$ by using matrix $P$, and write its pseudo-code.

**Solution:**

**Algorithm 4** SOLUTION($u, v, P$)

```
if u == v then
    return {u}
end if
return {u} + FIND-PATH(u, v, P) + {v}
```

**Algorithm 5** FIND-PATH($u, v, P$)

```
if P[u, v] == −1 then
    return ∅
end if
k = p[u, v]
return FIND-PATH(u, k, P) + {k} + FIND-PATH(k, v, P)
```

(b) (8 points) Consider the following matrix as matrix $P$ for graph $G$ with 6 vertices. What is the shortest path from vertex 1 to vertex 2 in graph $G$? What is the shortest path from vertex 5 to vertex 7 in graph $G$?

**Solution:** The shortest path from vertex 1 to 2 is $1, 4, 3, 5, 2$. The shortest path from vertex 5 to 7 is $5, 3, 6, 7$.

| $P$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | -1 | 5 | 4 | -1 | 4 | 4 | -1 |
| 2 | 5 | -1 | 5 | 5 | -1 | 5 | -1 |
| 3 | 4 | 5 | -1 | -1 | -1 | -1 | 6 |
| 4 | -1 | 5 | -1 | -1 | 3 | 3 | 1 |
| 5 | 4 | -1 | -1 | 3 | -1 | 3 | 6 |
| 6 | 4 | 5 | -1 | 3 | 3 | -1 | -1 |
| 7 | -1 | -1 | 6 | 1 | 6 | -1 | -1 |