Induction and Recurrence

Problem For every integer $n \geq 2$, prove $(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$

Problem For every integer $n \geq 1$, prove $3^n \geq n^2$.

Problem Prove any natural number can be written as sum of different powers of 2. For example $100 = 64 + 32 + 4$.

Problem The Fibonacci sequence is defined as follows: $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for every integer $n > 2$. Prove the following for Fibonacci sequence:

(a) $\sum_{i=1}^{n} F_i = F_{n+2} - 1$

(b) $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$

Problem n friends each have distinct news. They want to share the news such that all of them know all the news. Prove this can be achieved by making $2n - 4$ phone calls. (A phone call can be made between only two people and not more.)

Problem Let $A_1, A_2, \dots, A_m$ denote all non-empty subsets of $\{1, 2, \dots, n\}$. Let a_i denote product of the elements in A_i. Prove $\sum_{i=1}^{m} \frac{1}{a_i} = n$.

Problem n lines are drawn on the plane such that no three lines intersect at the same point. Prove that these lines divide the plane into $\frac{n \times (n+1)}{2} + 1$ regions.

Problem There are n teachers in a school. Teacher number i knows $i + 1$ students. Each teacher wants to select an assistant among students. No student can be assistant to more than one teacher. Prove that the teachers can chose their assistants in at least 2^n different ways.

Problem A group of people play the following game. At the beginning all the players close their eyes and a non-participating agent puts a sticker on each player's forhead. Some and at least one of these stickers are marked. No one can see their own sticker until the end of the game, and the players are not allowed to communicate. The game proceeds in rounds. At each round the players see everyone else's forheads. A player who is sure about her status (marked or unmarked) can shout it. If a player incorrectly guesses her status everybody would lose. If no one guesses, the game proceeds to the next round. The first player who correctly finds out her status wins. The players have no knowledge of the number of marked players. Prove that after k rounds where k is the number of marked players all the marked players win simultaneously. You can assume players are very smart and they only shout their status if they are sure about it. (Hint : Use induction on k.)

Problem Suppose we have a box with n balls. At each round, we pick a box with $m > 1$ balls in it, and divide the balls into two boxes with k and $m - k$ balls for some $1 \leq k < m$, and we write down $k \times (m - k)$. We continue this until each box has only one ball. Prove in any order that this procedure is done, sum of the numbers that we write down would be $\binom{n}{2}$.

Problem For any integer n, prove that 7^n circles with radius 1 can be drawn into a circle of radius 3^n so that no two circles cross. For $n = 1$, see figure 1

Problem Suppose a_n denotes the number of n digit numbers with only $1, 2, 3$ as their digits in which no two consecutive digits are 2. Give a recursive relation for a_n.

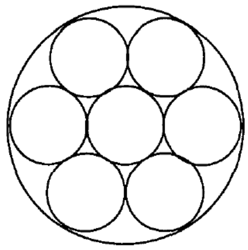

Figure 1:

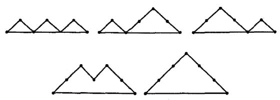

Figure 2:

Problem Let a_n be the number of ways that we can mark the cells of a $2 \times n$ table so that no two neighbor cells are marked. Find a recursive relation for a_n.

Problem n circles are drawn on a plane, each two of them collide and no three of them pass the same point. Let c_n denote the number of regions that these circles divide the plane into. Find a recursive relation for c_n and solve it.

Problem a) Suppose a particle is located at point $(0, 0)$. At each time if it is at point (x, y) it can either go to $(x + 1, y + 1)$ or $(x + 1, y - 1)$. However, the particle can never go to $y < 0$. Let C_n denote the number of ways that the particle can go to point $(2n, 0)$. For example if $n = 3$ you can see all the possible ways that the particle can go to the point $(6, 0)$ in figure 2. Therefore, $C_3 = 5$. Prove that $C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}$.

b) Mel and Mary have been candidated for an election. The votes are out and each of them have exactly n votes. Mel wants to put the votes in an order so that if someone goes through the votes and counts them one by one at no point she would have less votes than Mary. How many ways can she do this?