

Upper bound for harmonic sum

Let $n + 1$ be a power of 2.

$$\begin{aligned}
 H_n &= \underbrace{1}_{1} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{\frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{15}}_{\frac{1}{8}} + \cdots + \underbrace{\frac{1}{(n+1)/2} + \cdots + \frac{1}{n-1} + \frac{1}{n}}_{\frac{1}{(n+1)/2}} \\
 &\leq \underbrace{1}_{1} + \underbrace{\frac{1}{2} + \frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{\frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{8}}_{\frac{1}{8}} + \cdots + \underbrace{\frac{1}{(n+1)/2} + \cdots + \frac{1}{(n+1)/2} + \frac{1}{(n+1)/2}}_{\frac{1}{(n+1)/2}} \\
 &= \underbrace{\underbrace{1}_{1} + \underbrace{1}_{1} + \underbrace{1}_{1} + \underbrace{1}_{1} + \cdots + \underbrace{1}_{1}}_{k \text{ times}} \\
 &= k
 \end{aligned}$$

for some integer k .

The total number of terms n in the sum is

$$1 + 2 + 4 + \cdots + 2^{k-1}$$

So

$$n = \sum_{k=0}^{k-1} 2^k = 2^{(k-1)+1} - 1 = 2^k - 1$$

Adding 1 and then taking lg of both sides give

$$k = \lg(n + 1)$$

In conclusion, $\lg(n + 1)$ is an upper bound on the harmonic number H_n .