ASSIGNMENT 5

Due in class on Thursday, November 12.

1. Spindles and pancakes.
   (a) [2 points] Consider a map on density matrices that sends a state with Bloch vector \((x, y, z)\) to one with Bloch vector \((0, 0, z)\). Show that this map is a quantum operation.
   (b) [3 points] Consider a map on density matrices that sends a state with Bloch vector \((x, y, z)\) to one with Bloch vector \((x, y, 0)\). Is this map a quantum operation? Prove that your answer is correct.

2. Effect of noise on state distinguishability.
   Let \(|\psi\rangle = |0\rangle\) and \(|\phi\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle\).
   (a) [2 points] Recall that the depolarizing channel with parameter \(p \in [0, 1]\) is the quantum operation \(D_p(\rho) = (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)\) on a qubit state \(\rho\). Compute the action of the depolarizing channel on the states \(|\psi\rangle\) and \(|\phi\rangle\).
   (b) [4 points] Compute the trace distance between \(D_p(|\psi\rangle\langle\psi|)\) and \(D_p(|\phi\rangle\langle\phi|)\).
   (c) [1 point] Discuss how the depolarizing channel affects the distinguishability of quantum states.

3. Properties of (relative) entropy.
   (a) [3 points] The relative entropy (also known as the Kullback-Leibler divergence) of two probability distributions \(p\) and \(q\) is \(D(p\|q) = \sum_i p_i \log(p_i/q_i)\). Prove that for any probability distributions \(p\) and \(q\), we have \(D(p\|q) \geq 0\).
   (b) [3 points] The relative entropy of two quantum states \(\rho\) and \(\sigma\) is \(D(\rho\|\sigma) = \text{Tr}(\rho (\log \rho - \log \sigma))\). Let \(\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|\) and \(\sigma = \sum_i q_i |\phi_i\rangle\langle\phi_i|\) be spectral decompositions of \(\rho\) and \(\sigma\), respectively. Show that \(D(\rho\|\sigma) = -S(\rho) - \sum_{i,j} p_i |\langle\psi_i|\phi_j\rangle|^2 \log q_j\).
   (c) [3 points] Let \(r_i = \sum_j |\langle\psi_i|\phi_j\rangle|^2 q_j\). Show that \(D(\rho\|\sigma) \geq D(p\|r)\), and thereby conclude that \(D(\rho\|\sigma) \geq 0\).
   (d) [2 points] Use nonnegativity of the relative entropy to show that if \(\rho\) is supported on a space of dimension \(d\), then \(S(\rho) \leq \log d\).
   (e) [3 points] Use nonnegativity of the relative entropy to prove the subadditivity of entropy, i.e., \(S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)\). (Hint: consider the relative entropy of \(\rho_{AB}\) and \(\rho_A \otimes \rho_B\).)
4. **Entanglement concentration.**

(a) [3 points] Suppose Alice has $N$ qubits. For any $n \in \{0, 1, \ldots, N\}$, define a projector

$$
\Pi_n = \sum_{x: \text{wt}(x) = n} |x\rangle\langle x|
$$

onto the subspace of states with Hamming weight $n$, where the Hamming weight $\text{wt}(x)$ is the number of 1s in the bit string $x$. Describe a quantum circuit that performs the projective measurement $\{\Pi_n\}$. (Note that Alice should not measure any refinement of this measurement, so superpositions of different $x$ with the same Hamming weight remain coherent.)

(b) [3 points] Suppose Alice and Bob share $N$ copies of the state $|\psi\rangle = \alpha |00\rangle + \beta |11\rangle$, i.e., they have the state $|\psi\rangle^\otimes N$. Show that if Alice and Bob each perform the measurement from part (a), they will get the same outcome $n$, and show that the remaining state is equivalent, up to local unitary operations, to the maximally entangled state

$$
\binom{N}{n}^{-1/2} \sum_{j=0}^{(N)-1} |j\rangle_A |j\rangle_B.
$$

(c) [4 points] Let $\rho_A = \text{Tr}_B |\psi\rangle\langle \psi|$. Show that for large $N$, the most likely value of $n$ in the measurement from part (b) satisfies

$$
\log \binom{N}{n} \approx NS(\rho_A).
$$

(In fact, for large $N$, the distribution is very highly peaked around this value of $n$. In other words, using only local operations and classical communication, Alice and Bob can convert $N$ partially entangled states into about $NS(\rho_A)$ maximally entangled states with high probability.)