Instructions:

- Print the exam (single-sided printing is recommended).
- Print your full name clearly above.
- You may take the exam any time before the submission deadline. Once you begin the exam, you have three hours to complete it, with no breaks. Find a place where you can work without interruption.
- You are responsible for monitoring the time and stopping after three hours have elapsed. When you begin the exam, record your start time above. When you stop working on the exam, record your end time.
- The total number of exam pages is 11, not including this cover page. Page numbers are at the bottom of each page. Please check that you have all pages before beginning.
- You should complete this exam using only the exam paper and writing implements. You may not use any reference material or electronic devices (other than a clock or timer). You may not discuss the exam with anyone except the instructor and the TA.
- Write your name at the top of each page of the exam.
- The number of points allotted to each part is indicated in square brackets.
- Answer each question in the space provided. If you need more space, you can use the back of the previous page. Two additional pages have been provided at the end of the exam for rough work. Please write clearly and give full details of all calculations and proofs.
- You may submit the exam in either of two ways:
  - Clearly scan all pages of your exam, front and back, and email the output as a PDF file to amchilds@umd.edu. The time spent scanning your exam does not count toward the three hours of exam time, but your email must arrive by the deadline of noon on Thursday, December 17.
  - During normal business hours, bring the hardcopy of your exam to Javiera Caceres (QuICS Coordinator) in CSS 3103, by noon on Thursday, December 17.
- Sign below to indicate your understanding of the exam policies.

I have read and agree to follow the above instructions.

Signed: ______________________________
1. Tripartite entanglement.

[3 points] A state with any number of subsystems is called entangled if it cannot be written as a tensor product of separate quantum states for the subsystems. Is the three-qubit state

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

an entangled state? Prove that your answer is correct.
2. **Partial measurement.**

Suppose we measure the first qubit of the state

\[
\frac{1}{\sqrt{3}}(|00\rangle - |01\rangle + i|10\rangle)
\]

in the basis \{|+, -\}\}, where \(|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\).

(a) [2 points] What is the probability of getting the outcome “+”?

(b) [2 points] If the outcome is “+”, what is the post-measurement state of the second qubit?
3. *Distinguishing states with local operations and classical communication.*

Suppose Alice and Bob are each given one qubit of a two-qubit state that is promised to be either $|\psi\rangle$ or $|\phi\rangle$, for some fixed states $|\psi\rangle, |\phi\rangle$. Working together, their goal is to tell which state they have using only local measurements. They can each apply some one-qubit gate to their part of the state and then measure in the computational basis. They can then compare their measurement results by exchanging classical information, but they cannot send qubits or perform any two-qubit gates. For each pair of states, either give such a procedure that Alice and Bob can use to distinguish the states perfectly, or explain why this is not possible.

(a) [1 point] $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

(b) [1 point] $|\psi\rangle = |00\rangle$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

(c) [3 points] $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
4. *Circuits and universality.*

Let $U$ denote the two-qubit unitary operation implemented by the following quantum circuit:

\[ 
\begin{array}{c}
\bullet \\
\rightarrow \\
\rightarrow \\
\end{array}
\]

(a) [2 points] Write $U$ using Dirac notation in the computational basis.

(b) [2 points] Let $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$. Prove that the gate set \{\(U, H, T\)\} is universal. (You may appeal to any universality result discussed in class.)
5. \textit{Deutsch’s problem modulo }n.\textit{ }

Let \( n \) be a positive integer. Suppose you are given a black-box function \( f : \{0,1,\ldots,n-1\} \to \{0,1,\ldots,n-1\} \). A quantum black box \( U_f \) for \( f \) acts as \( U_f|x\rangle|z\rangle = |x\rangle|z + f(x) \mod n\rangle \).

(a) \([2 \text{ points}]\) If \( f(x) \) is independent of \( x \), we say \( f \) is \textit{constant}; if \( f \) is a bijection, we say it is \textit{balanced}. We are promised that \( f \) is either constant or balanced. What is the minimum number of classical queries required to determine with certainty which is the case?

(b) \([3 \text{ points}]\) Find a fixed quantum state \( |\psi\rangle \in \mathbb{C}^n \), independent of \( f \), such that for all \( x \in \{0,1,\ldots,n-1\} \), \( U_f|x\rangle|\psi\rangle = e^{2\pi if(x)/n}|x\rangle|\psi\rangle \).
(c) [6 points] Describe a quantum algorithm that determines whether $f$ is constant or balanced using only one query. Prove that your algorithm works.
6. The Steane code and fault tolerance.

The Steane code encodes one logical qubit into seven physical qubits, with the logical states

$$|0_L\rangle = \frac{1}{2\sqrt{2}}(|0000000\rangle + |1111000\rangle + |1100110\rangle + |1010101\rangle$$

$$+ |0011110\rangle + |0101101\rangle + |0110011\rangle + |1001011\rangle)$$

$$|1_L\rangle = \frac{1}{2\sqrt{2}}(|0000111\rangle + |1111111\rangle + |1100001\rangle + |1010010\rangle$$

$$+ |0011001\rangle + |0101010\rangle + |0110100\rangle + |1001100\rangle).$$

It has the stabilizer operators

$$S_1 = X\otimes X\otimes X\otimes I\otimes I\otimes I$$
$$S_2 = X\otimes X\otimes I\otimes I\otimes X\otimes X$$
$$S_3 = X\otimes I\otimes X\otimes I\otimes X\otimes I$$
$$S_4 = Z\otimes Z\otimes Z\otimes Z\otimes I\otimes I\otimes I$$
$$S_5 = Z\otimes Z\otimes I\otimes I\otimes Z\otimes Z\otimes I$$
$$S_6 = Z\otimes I\otimes Z\otimes I\otimes Z\otimes I\otimes Z$$

and can correct any single-qubit error. (Recall that $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.)

(a) [3 points] Determine the syndrome for each of the following errors:

- $X$ error on the first qubit

- $Z$ error on the second qubit

- $Y$ error on the third qubit
(b) [2 points] Show that $X \otimes X \otimes X \otimes X \otimes X \otimes X$ acts as a logical $X$ operator.

(c) [2 points] Show that $Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z$ acts as a logical $Z$ operator.

(d) [1 point] Recall that we say that an implementation of a logical gate is fault tolerant if a single-qubit error applied at any stage of the circuit results in at most one single-qubit error on each logical qubit of the output. It can be shown (for example, by direct calculation) that a controlled-not gate between two logical qubits encoded with the Steane code can be implemented by applying a controlled-not gate from the $j$th physical qubit of the first logical qubit to the $j$th corresponding physical qubit of the second logical qubit (for each $j = 1, 2, \ldots, 7$). Explain why this implementation is fault tolerant.
7. Private quantum channel.

(a) [3 points] Let $\rho$ be the density matrix of a qubit. Let $\mathcal{E}(\rho) = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$. Prove that for any $\rho$, $\mathcal{E}(\rho) = \frac{1}{4}I$.

(b) [2 points] Suppose Alice and Bob privately share two random bits $b_x, b_z \in \{0, 1\}$, and Alice would like to send a qubit described by the density matrix $\rho$ to Bob. Alice performs the operation $X^{b_x}Z^{b_z}$ and sends the resulting state to Bob. Explain why an eavesdropper who intercepts the transmission cannot obtain any information about $\rho$.

(c) [1 point] What operation should Bob perform to recover the state $\rho$ (assuming the state was not altered in transit)?