

These are examples of proofs used in cmsc250. These proofs tend to be very detailed. You can be a little looser.

## General Comments

# Proofs by Mathematical Induction

- If a proof is by Weak Induction the Induction Hypothesis must reflect that. I.e., you may NOT write the Strong Induction Hypothesis.
- The Inductive Step MUST explicitly state where the Inductive Hypothesis is used. (Something like “by IH” is good.)

# Example Proof by Weak Induction

*Theorem.* For  $n \geq 1$ ,  $\sum_{i=1}^n 4i - 2 = 2n^2$ .

**BASE CASE:** Let  $n = 1$ . The summation gives

$$\sum_{i=1}^n 4i - 2 = \sum_{i=1}^1 4i - 2 = 4 \cdot 1 - 2 = 2.$$

The formula gives

$$2n^2 = 2 \cdot 1^2 = 2.$$

The two values are the same.

- **INDUCTIVE HYPOTHESIS** [*Choice I: From  $n - 1$  to  $n$ :*]

Assume that the theorem holds for  $n - 1$  (for arbitrary  $n > 1$ ). Then

$$\sum_{i=1}^{n-1} 4i - 2 = 2(n - 1)^2.$$

[*It is optional to simplify the right side. If not, it will have to be done inside the Induction Step.*]

- **INDUCTIVE STEP:** [*Choice Ia: Start with the sum we care about.*]

$$\begin{aligned} \sum_{i=1}^n 4i - 2 &= \sum_{i=1}^{n-1} i + (4n - 2) && \text{by splitting sum} \\ &= 2(n - 1)^2 + (4n - 2) && \text{by IH} \\ &= 2(n^2 - 2n + 1) + (4n - 2) && \text{by algebra} \\ &= 2n^2. && \text{by algebra} \end{aligned}$$

So the theorem holds for  $n$ .

- **INDUCTIVE STEP:** [*Choice Ib: Start with the induction hypothesis.*]

$$\begin{aligned} \sum_{i=1}^{n-1} 4i - 2 &= 2(n - 1)^2 && \text{by IH} \\ \sum_{i=1}^{n-1} 4i - 2 + (4n - 2) &= 2(n - 1)^2 + (4n - 2) && \text{adding } 4n - 2 \text{ to both sides} \\ \sum_{i=1}^n 4i - 2 &= 2(n^2 - 2n + 1) + (4n - 2) && \text{merging the sum on left side} \\ & && \dots \text{ and algebra on the right side} \\ &= 2n^2. && \text{by algebra on the right side} \end{aligned}$$

So the theorem holds for  $n$ .

- **INDUCTIVE HYPOTHESIS:** [*Choice II: From  $n$  to  $n + 1$* ]

Assume that the theorem holds for arbitrary  $n \geq 1$ . Then

$$\sum_{i=1}^n 4i - 2 = 2n^2.$$

[The following is optional (but often useful). If you do not do this, you will struggle to make the right side have the correct form with  $n$  replaced by  $n + 1$ . In principle, it should be inside the Inductive Step. At the very least, it must be clearly distinct from the Inductive Hypothesis.]

NEED TO SHOW:

$$\sum_{i=1}^{n+1} 4i - 2 = 2(n+1)^2 = 2(n^2 + 2n + 1) = 2n^2 + 4n + 2 .$$

– **INDUCTIVE STEP:** [Choice IIa: Start with the sum we care about.]

$$\begin{aligned} \sum_{i=1}^{n+1} 4i - 2 &= \sum_{i=1}^n i + (4(n+1) - 2) && \text{by splitting sum} \\ &= 2n^2 + (4(n+1) - 2) && \text{by IH} \\ &= 2n^2 + (4n + 2) && \text{by algebra} \\ &= 2n^2 + 4n + 2 . && \text{by algebra} \end{aligned}$$

This is what we needed to prove, so the theorem holds for  $n + 1$ .

– **INDUCTIVE STEP:** [Choice IIb: Start with the induction hypothesis.]

$$\begin{aligned} \sum_{i=1}^n 4i - 2 &= 2n^2 && \text{by IH} \\ \sum_{i=1}^n 4i - 2 + (4(n+1) - 2) &= 2n^2 + (4(n+1) - 2) && \text{adding } 4(n+1) - 2 \text{ to both sides} \\ \sum_{i=1}^{n+1} 4i - 2 &= 2n^2 + 4n + 2 && \text{merging sum on the left side} \\ &&& \dots \text{ and algebra on the right side} \end{aligned}$$

This is what we needed to prove, so the theorem holds for  $n + 1$ .

## Example Proof by Strong Induction

**BASE CASE:** [Same as for Weak Induction.]

- **INDUCTIVE HYPOTHESIS:** [Choice I: Assume true for less than  $n$ ]

(Assume that for arbitrary  $n > 1$ , the theorem holds for all  $k$  such that  $1 \leq k \leq n - 1$ .)

Assume that for arbitrary  $n > 1$ , for all  $k$  such that  $1 \leq k \leq n - 1$  that

$$\sum_{i=1}^k 4i - 2 = 2k^2 .$$

- **INDUCTIVE HYPOTHESIS:** [Choice II: Assume true for less than  $n + 1$ ]

(Assume that for arbitrary  $n \geq 1$  the theorem holds for all  $k$  such that  $1 \leq k \leq n$ .)

Assume that for arbitrary  $n > 1$ , for all  $k$  such that  $1 \leq k \leq n$  that

$$\sum_{i=1}^k 4i - 2 = 2k^2 .$$

**INDUCTIVE STEP:** [And now a brilliant proof that somehow uses strong induction.]

# Constructive Induction

*[We do this proof only one way, but any of the styles is fine.]*

Guess that the answer is quadratic, so it has form  $an^2 + bn + c$ . We will derive the constants  $a, b, c$  while proving it by Mathematical Induction.

**BASE CASE:** Let  $n = 1$ . The summation gives

$$\sum_{i=1}^n 4i - 2 = \sum_{i=1}^1 4i - 2 = 4 \cdot 1 - 2 = 2 .$$

The formula gives

$$an^2 + bn + c = a \cdot 1^2 + b \cdot 1 + c = a + b + c .$$

So, we need  $a + b + c = 2$ .

**INDUCTIVE HYPOTHESIS:**

Assume that the theorem holds for  $n - 1$  (for arbitrary  $n > 1$ ). Then

$$\sum_{i=1}^{n-1} 4i - 2 = a(n - 1)^2 + b(n - 1) + c .$$

*[Again, it is optional to simplify the right side.]*

**INDUCTIVE STEP:**

$$\begin{aligned} \sum_{i=1}^n 4i - 2 &= \sum_{i=1}^{n-1} 4i - 2 + (4n - 2) && \text{by splitting sum} \\ &= a(n - 1)^2 + b(n - 1) + c + (4n - 2) && \text{by IH} \\ &= a(n^2 - 2n + 1) + b(n - 1) + c + (4n - 2) && \text{by algebra} \\ &= an^2 + (-2a + b + 4)n + a - b + c - 2 && \text{by algebra} \\ &= an^2 + bn + c . && \text{to make the induction work} \end{aligned}$$

The coefficients on each of the powers have to match. This leads to the three simultaneous equations:

$$\begin{aligned} b &= -2a + b + 4 \\ c &= a - b + c - 2 \\ 2 &= a + b + c \quad \text{from the base case} \end{aligned}$$

The first equation gives  $a = 2$ , then the second gives  $b = 0$ , and finally the third gives  $c = 0$ .

# Constructive Induction (Another Example)

Problem: Find an upper bound on  $F_n$  in the recurrence

$$F_n = F_{n-1} + F_{n-2}$$

where  $F_0 = F_1 = 1$ .

Guess that the answer is exponential, so  $F_n \leq ab^n$ . We will derive the constants  $a, b$  while proving it by Mathematical Induction.

**BASE CASES:** Let  $n = 0$ . By definition

$$F_n = F_0 = 1$$

The formula gives

$$F_n \leq ab^n = ab^0 = a$$

So,  $a \geq 1$ .

Let  $n = 1$ . By definition

$$F_n = F_1 = 1$$

The formula gives

$$F_n \leq ab^n = ab^1 = ab$$

So,  $ab \geq 1$ .

**INDUCTIVE HYPOTHESIS:**

Assume that for arbitrary  $n > 1$ , for all  $k$  such that  $1 \leq k \leq n - 1$  that  $F_k \leq ab^k$ .

**INDUCTIVE STEP:**

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} && \text{by definition} \\ &\leq ab^{n-1} + ab^{n-2} && \text{by IH} \\ &\leq ab^n && \text{to make the induction work} \end{aligned}$$

Thus we need to solve

$$ab^{n-1} + ab^{n-2} \leq ab^n .$$

or

$$b^2 - b - 1 \geq 0 .$$

By the quadratic formula, we get

$$b \geq \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

Only the positive value can hold. Also, we would like the smallest possible value for  $b$ . So we choose

$$b = \frac{1 + \sqrt{5}}{2}$$

From the base cases we get  $a \geq 1$  (since the other condition is weaker), and now we would like the smallest possible value for  $a$ . So we choose  $a = 1$ . This gives

$$F_n \leq \left( \frac{1 + \sqrt{5}}{2} \right)^n$$

# Catalan Numbers

**Theorem.** For  $n \geq 1$ ,  $\frac{(2n)!}{n!(n+1)!} \geq \frac{4^n}{(n+1)^2}$ .

*Proof.* by Mathematical Induction.

**BASE CASE:** Easy.

**INDUCTION HYPOTHESIS:** Assume true for  $n - 1$ :

$$\frac{(2(n-1))!}{(n-1)!n!} \geq \frac{4^{n-1}}{n^2}.$$

**INDUCTION STEP:** *Alternative I*

$$\begin{aligned} \frac{(2n)!}{n!(n+1)!} &= \frac{(2n)(2n-1)}{(n-1)n} \frac{(2(n-1))!}{(n-1)!n!} \\ &\geq \frac{(2n)(2n-1)}{n(n+1)} \frac{4^{n-1}}{n^2} && \text{by IH} \\ &= \frac{(2n)(2n-1)}{n(n+1)} \frac{(n+1)^2}{4n^2} \frac{4n^2}{(n+1)^2} \frac{4^{n-1}}{n^2} \\ &= \frac{(2n)(2n-1)}{(n-1)n} \frac{(n+1)^2}{4n^2} \frac{4^n}{(n+1)^2} \\ &= \frac{(1-1/2n)(1+1/n)}{1-1/n} \frac{4^n}{(n+1)^2} \\ &= \frac{1+1/(2n)-1/(2n^2)}{1-1/n} \frac{4^n}{(n+1)^2} \\ &\geq \frac{4^n}{(n+1)^2}. \end{aligned}$$

**INDUCTION STEP:** *Alternative II*

$$\begin{aligned} \frac{4^n}{(n+1)^2} &= \frac{4n^2}{(n+1)^2} \frac{4^{n-1}}{n^2} \\ &\leq \frac{4n^2}{(n+1)^2} \frac{(2(n-1))!}{(n-1)!n!} && \text{by IH} \\ &= \frac{4n^2}{(n+1)^2} \frac{n(n+1)}{(2n)(2n-1)} \frac{(2n)(2n-1)}{n(n+1)} \frac{(2n-1)!}{(n-1)!n!} \\ &= \frac{1}{(1+1/n)(1-1/(2n))} \frac{(2n)!}{n!(n+1)!} \\ &= \frac{1}{(1+1/(2n)-1/(2n^2))} \frac{(2n)!}{n!(n+1)!} \\ &\leq \frac{(2n)!}{n!(n+1)!}. \end{aligned}$$

□