Leximin Allocation in the Real World

Authors: David Kurokawa, Ariel D. Procaccia, and Nisarg Shah

Presentation by: Hadi Yami

- The authors of the paper has a fair allocation website (spliddit.org)
- It offers fair allocation solutions for the division of goods, rent, and credit.
- Herve Moulin, an economist, suggested them to let people ask their own fair allocation problems.
- The problem of this paper is one of these questions.

The Problem

- “public school facilities should be shared fairly among all public schools pupils, including those in charter schools.”
- Unfortunately, the law does not elaborate on what “fairly” means.

Formal Description of the Problem

- We have n charter schools looking for classrooms in m public schools.
- We call charter schools as agents, and public schools as facilities.
- We want to design a mechanism for assigning the agents to the facilities.
- Each facility f has a capacity c_f , which is the number of units available at the facility. (the number of classrooms here.)

Formal Description of the Problem(cont)

- The preferences of agent i: (d_i , F_i)
- Demands: d_i , and acceptable facilities: F_i
- Reason : geographical situations or …
- The agent has utility 1 if it receives d_i units from any single facility, and 0 otherwise.
- An allocation A is feasible if it respects the capacity constraints at each facility.

- We are looking for a mechanism which is:
- Proportional: u_i $(A_i) \geq 1/n$ for all agents.
- Envy-freeness: $u_i(A_i) \geq u_i(A_j)$ for all (i, j) of agents.
- Pareto optimal
- Strategyproof

Candidate Mechanism 1

- A simple randomized mechanism that allocates all available units at all facilities to each agent with probability 1/n
- Proportional: Yes
- Envy-free: Yes
- Strategyproof: Yes
- Pareto optimal: No!

Candidate Mechanism 2

- A mechanism that always returns a deterministic allocation maximizing the number of units allocated.
- Pareto Optimal: Yes
- Proportional: No
- Envy-free: No
- Strategyproof: No
- Example: one facility with ($c_1 = 4$), two agents with demands ($d_1 = 3$, $d_2 = 2$).

The Leximin Mechanism

- We try to maximize the minimum utility, then the second minimum utility, and so on so forth.
- Example: two facilities, $c_a = 1$, $c_b = 2$.
- 4 agents: $d_1 = 1$, $d_2 = 1$, $d_3 = 2$, $d_4 = 1$
- $F_1 = \{a\}$, $F_2 = F_3 = \{b\}$, $F_4 = \{a, b\}$
- The Leximin Allocation: ½ (a : {1}, b: {2, 4}) + ¼ (a : {1}, b : {3}) + ¼ (a: {4}, b : {3}).
- Utility ½ to agents 2 and 3, and utility ¾ to agents 1 and 4

Properties

- Proportional: Yes, why?
- Pareto optimal: Yes, why?
- Envy-freeness: Yes
- Group Strategyproofness: Yes
- (stronger than strategyproofness) It requires that if a subset of agents simultaneously report false preferences, at least one of the agents in the subset must not be strictly better off.

Stronger than groupstrategyproofness?

- Group strategyproofness is strong but in a stronger requirement a group of manipulators should not be able to report false preferences that would lead to all manipulators being weakly happier and at least one manipulator being strongly happier.
- A counter example exists for the leximin allocation ☹

Lorenz Dominance

- Let u^k and v^k denote the k-th lowest utility in allocations A and B, We say that allocation A (weakly) Lorenz-dominates allocation B if $\sum_{i=1}^{k} u^i \geq \sum_{i=1}^{k} v^i$ for k ∈ {1, … , n}.

- In our setting?

Lorenz Dominance in Our Setting

- Lorenz Dominance condition:
  $\sum_{i=1}^{k} u^i \geq \sum_{i=1}^{k} v^i$ for k ∈ {1, … , n}
- In our setting there may not exist an allocation that weakly Lorenz-dominates every other allocation.
- Example: a single facility with $c_1 = 3$, and 4 agents with $d_1 = 3$, $d_2 = d_3 = d_4 = 1$
- The allocation A must achieve the maximum possible lowest utility.
- Hence, A should assign agent 1 to the facility with probability 0.5 and assign the remaining agents to the facility with the remaining probability 0.5
- The sum of three lowest utilities under A is 1.5
- However, for the allocation that assigns agents 2 through 4 to the facility with probability 1, the sum of the 3 lowest utilities is 2!

Two Quantitative Notions of efficiency

- 1 - Maximizing the number of allocated units
- (An example in slide 8) – it was not proportional, envy-free, and strategyproof
- 2 - Maximizing the number of satisfied agents
- Example: a single facility with $c_1 = 2$, and four agents with $d_1 = d_2 = d_3 = 1$, $d_4 = 2$
- We must allocate a single unit to two agents in {1, 2, 3}
- But this allocation is not envy-free and proportional! ☹

Efficiency of Allocations in the Support of the Leximin Allocation

- Example: Two facilities with c_1 = k, c_2 = k^2, and k+4 agents with following preferences:
- (d_i , F_i) = (1, {1}) if i ∈ {1, …, k}
- (d_i , F_i) = (k, {1}) if i = k +1 or k + 2
- (d_i , F_i) = (1, {2}) if i = k + 3
- (d_i , F_i) = (k^2, {2}) if i = k + 4
- A maximum of k + 1 agents can be satisfied, and a maximum of k + k^2 units can be allocated.
- Leximin allocation:
- agents 1 through k + 2 should be assigned to facility 1 with prob: 1/3 each
- Agents k + 3 and k + 4 to facilitiy 2 with prob: ½ each
- Worst case Scenario: we assign agents k+1 and k+2 to facility 1, and agent k +3 to facility 2.
- The number of satisfied agents: 2/(k+1) fraction of the optimum
- The number of allocated units: (k+1) / (k + k^2) = 1/k fraction of the optimum
- Both approximation ratios converge to 0 as k goes to infinity.

Proportionality and the Number of Satisfied agents

- A single facility with c_1 = k, and k + k^2 agents, k of which require 1 unit each, and the other k^2 agents require all k units each.
- Any proportional mechanism must allocate the k units to each of the k^2 agents with prob at least 1/(k + k^2).
- This mechanism satisfies a single agent with prob at least k^2/(k + k^2), and at most k agents with the remaining prob
- The expected number of satisfied agents is at most: k^2/(k + k^2) + k . k/(k + k^2) ≤ 2
- A maximum of k agents could be satisfied simultaneously.
- Any proportional mechanism(including the leximin mechanism) achieves an approximation ratio of 2/k for the number of satisfied agents. This ratio goes to 0 as k goes to infinity

Leximin Allocation and the Number of Allocated Units

- **Theorem**: The expected number of units allocated by the leximin mechanism 4-approximates the maximum number of units that can be allocated simultaneously by any non-wasteful allocation (in the worst case over instances).
- **Conjecture**: 2-approximation??
- We cannot do better than 2-approximation

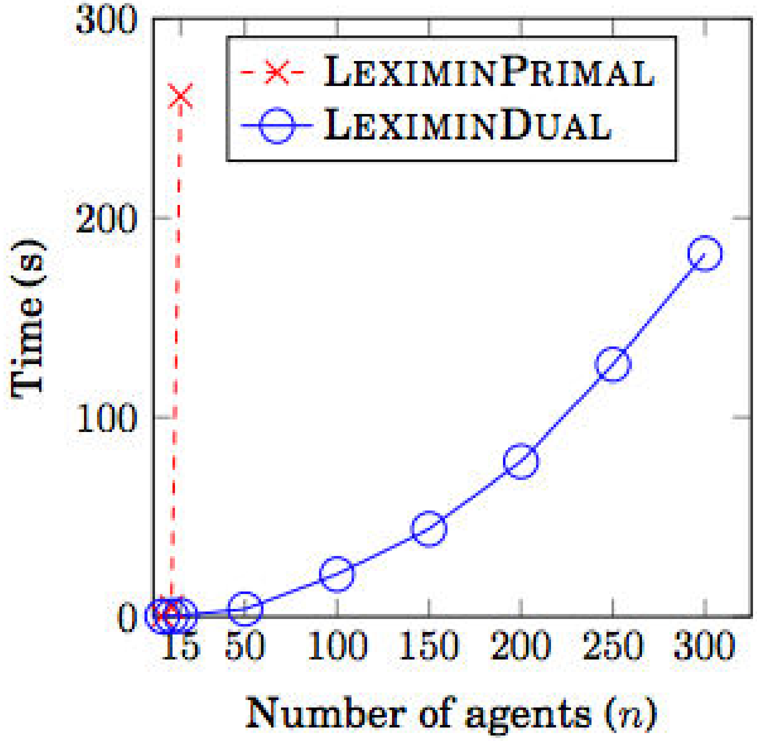
Implementation

Data: Demands $\{(d_i, F_i)\}_{i \in N}$, Capacities $\{c_j\}_{j \in M}$
Result: The Leximin Allocation A
Solve FEASIBILITYILP for each $S \subseteq N$, and let $\mathcal{S} \leftarrow$ the set of maximal feasible subsets of N;
For each $S \in \mathcal{S}$, $A_S \leftarrow$ the assignment returned by FEASIBILITYILP on S;
$R = N$;
$p_i^* = 0, \forall i \in N$;
do
  $(M, \{p_i\}_{i \in R}, \{x_S\}_{S \in \mathcal{S}}) \leftarrow$ Strictly complementary solution to PRIMALLP in the box below;
  $p_i^* = M, \forall i \in R : p_i = M$;
  $R = R \setminus \{i \in N | p_i = M\}$;
  if $R = \emptyset$ **then**
    return the randomized allocation where A_S is executed with probability x_S for each $S \in \mathcal{S}$;
  end
while $R \neq \emptyset$;

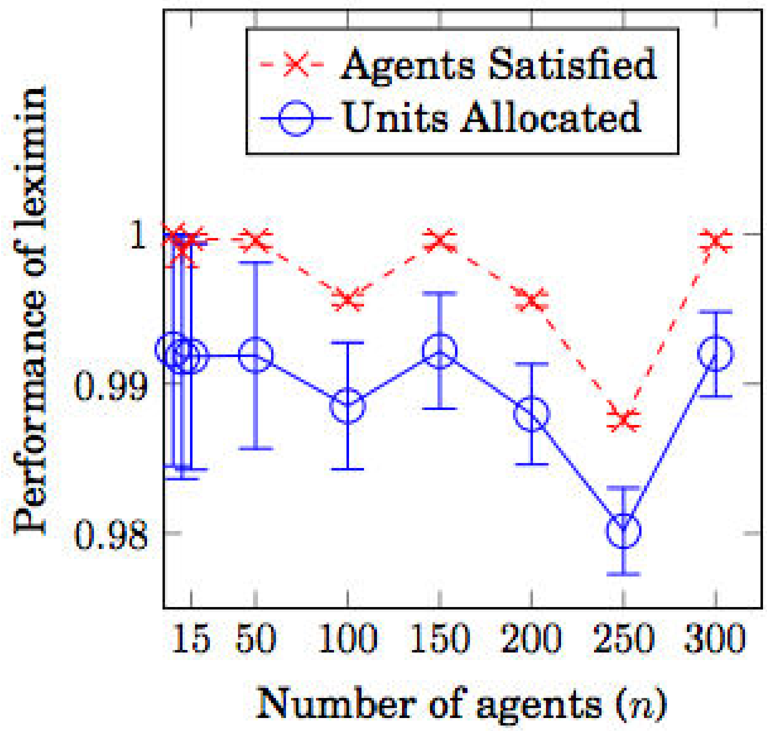
PRIMALLP:
Maximize M subject to $p_i \geq M, \forall i \in R$ $p_i = p_i^*, \forall i \in N \setminus R$ $p_i = \sum_{S \in \mathcal{S}, i \in S} x_S, \forall i \in N$ $\sum_{S \in \mathcal{S}} x_S = 1$ $x_S \geq 0, \forall S \in \mathcal{S}$

FEASIBILITYILP:
$\sum_{f \in F_i} y_{i,f} \geq 1, \forall i \in S$ $\sum_{i \in S : f \in F_i} d_i \cdot y_{i,f} \leq c_f, \forall f \in M$ $y_{i,f} \in \{0, 1\}, \forall i \in S, f \in F_i$

Results



300
200
100
0
Time (s)
LeximinPrimal
LeximinDual
15 50 100 150 200 250 300
Number of agents (n)



Performance of leximin
1
0.99
0.98
Agents Satisfied
Units Allocated
15 50 100 150 200 250 300
Number of agents (n)

Thank you.