ASSIGNMENT 5 CMSC 858K (Fall 2016)

Due in class on Tuesday, Nov 22.

1. *Effect of noise on state distinguishability.*

   Continuing Problem 4 from Assignment 1, and Problem 6 from Assignment 4, consider the task of distinguishing $|\psi\rangle = |0\rangle$ and $|\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$. Here however we do not assume these states are perfectly prepared; we model the states we obtain as having gone through a depolarizing channel.

   (a) [2 points] Recall that the depolarizing channel with parameter $p \in [0,1]$ is given by the Kraus form $\mathcal{D}_p(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$ on a qubit state ρ. Compute the action of the depolarizing channel on the states $|\psi\rangle$ and $|\phi\rangle$.

   (b) [3 points] Compute the trace distance between $\mathcal{D}_p(|\psi\rangle\langle\psi|)$ and $\mathcal{D}_p(|\phi\rangle\langle\phi|)$.

   (c) [2 points] Discuss how the depolarizing channel affects the distinguishability of quantum states. Specifically, what is the effect of depolarization on the optimal POVM computed in Problem 6(c) of Assignment 4.

   (d) [1 point] If $p > 0$ is it possible to find a POVM that *unambiguously* distinguishes either of the states $\mathcal{D}_p(|\psi\rangle\langle\psi|)$ or $\mathcal{D}_p(|\phi\rangle\langle\phi|)$ with nonzero probability?

2. *Quantum process tomography.*

   In this problem we focus on quantum channels $\mathcal{C}$ that map qubit density operators on $\mathbb{C}^2$ back into densities on $\mathbb{C}^2$. We write $L(\mathfrak{H})$ for the space of operators on the Hilbert space $\mathfrak{H}$. The process matrix form for a channel is $\mathcal{C}(\rho) = \sum_{ab}\chi_{ab}E_a\rho E_b$ where $\{E_a\}_{a=1}^{(\dim\mathfrak{H})^2}$ is a vector space basis of $L(\mathfrak{H})$. So for qubit channels, (χ_{ab}) is a 4×4 matrix. You may assume that the process matrix (χ_{ab}) is uniquely determined by the channel.

   (a) [2 points] Recall that the "Hilbert-Schmidt" inner product of operators (on a 2-dimensional Hilbert space) is $\langle A, B\rangle = \frac{1}{2}\operatorname{Tr}(A^\dagger B)$. Prove the "Pauli basis" $\{\mathbb{1}, X, Y, Z\}$ is an orthonormal basis of $L(\mathbb{C}^2)$ with respect to the Hilbert-Schmidt inner product. Also show that in Pauli basis, the process matrix for any quantum channel on qubits is itself Hermitian and trace one.

   (b) [2 points] Recall the amplitude damping channel $\mathcal{A}_\gamma$ is defined through its Kraus operators $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$ and $E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$. Compute the process matrices for the depolarizing channel $\mathcal{D}_p$ and the amplitude damping channel $\mathcal{A}_\gamma$ in the Pauli basis.

   (c) [1 point] (State tomography) Suppose we have a device that outputs a qubit in an unknown state ρ. Running the device many times, and measuring the output qubit in one of the X-, Y-, or Z-directions, we obtain statistics $\langle X\rangle_\rho = \operatorname{Tr}(X\rho) = 0.66$, $\langle Y\rangle_\rho = \operatorname{Tr}(Y\rho) = -0.02$, and $\langle Z\rangle_\rho = \operatorname{Tr}(Z\rho) = 0.24$. What is our empirical estimate for ρ?

   (d) [2 points] Suppose we have a qubit device the implements some process described by an unknown channel $\mathcal{C}$. Fix a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with complex coefficients. Theoretically, we can write $\mathcal{C}(M) = \sum_{ab}\chi_{ab}E_aME_b$. Explain how we could empirically estimate $\mathcal{C}(M)$ by running the device on specially prepared quantum states. How does this lead to a method for estimating the unknown process matrix (χ_{ab}) (you needn't write out the estimate itself, just explain how to compute it).

3. *Quantum Hamming bound.*

   For a quantum code $\mathfrak{C} \subset \mathfrak{H}$, here we examine a relationship between n (the number of qubits in $\mathfrak{H}$), k (the number of logical/encoded qubits of $\mathfrak{C}$), and t (the maximum number of qubits involved in an error). You may freely use the quantum error correction condition: a set of errors $\{E_j\}$ is correctable if $\Pi_{\mathfrak{C}} E_j^\dagger E_k \Pi_{\mathfrak{C}} = \alpha_{jk} \Pi_{\mathfrak{C}}$. A code is *non-degenerate* (with respect to the errors $\{E_j\}$) if in the quantum error correction condition the matrix (α_{jk}) is invertible.

   (a) [1 point] Prove: if the code $\mathfrak{C}$ can correct a set of error operators $\{E_j\}$, then it can correct any error in their linear span, $F = \sum_j c_j E_j$.

   (b) [2 points] Prove: if the code $\mathfrak{C}$ can correct any Pauli error of weight at most t, then it can correct *any* error that involves t qubits. How many such Pauli error operators are there?

   (c) [3 points] Prove: if the code $\mathfrak{C}$ is non-degenerate for a set of errors $\{E_j\}$, then there is a basis $\{\hat{E}_j\}$ for span$\{E_j\}$ with the properties for any $|\psi\rangle \in \mathfrak{C}$, (i) $\hat{E}_j |\psi\rangle \neq \mathbf{0}$ (i.e. $\hat{E}_j$ is injective on $\mathfrak{C}$), and (ii) $\hat{E}_j |\psi\rangle$ is orthogonal to $\hat{E}_k |\psi\rangle$ whenever $j \neq k$. Use this to prove the quantum Hamming bound for non-degenerate codes: $\sum_{j=0}^{t} \binom{n}{j} 3^j 2^k \leq 2^n$.

   (d) [1 point] According to this bound, what is the smallest number of qubits n for which there *may* be a nondegenerate quantum code that encodes one qubit and corrects any single qubit error?

4. *Three-bit repetition codes under damping channels.*

   Recall the syndrome measurements of the 3-bit repetition code are Z_1Z_2 and Z_2Z_3. For the rotated 3-bit repetition code they are X_1X_2 and X_2X_3.

   (a) [3 points] Consider encoding a general qubit state $\alpha|0\rangle + \beta|1\rangle$ into the 3-bit repetition code as $\alpha|000\rangle + \beta|111\rangle$. Compute the effect of the amplitude damping channel $\mathcal{A}_\gamma$ on the first qubit, and find the probabilities for each possible syndrome measurement? What is the state of the system after error correction (treated as an ensemble over the possible measurement results)? What is the fidelity of the input and output states?

   (b) [3 points] Recall the phase damping channel $\mathcal{P}_\lambda$ has Kraus operators $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}$ and $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$. Now consider encoding the qubit into the rotated 3-bit repetition code as $\alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$. Compute the effect of the phase damping channel $\mathcal{P}_\lambda$ on the first qubit, and find the probabilities for each possible syndrome measurement? What is the state of the system after error correction (treated as an ensemble over the possible measurement results)? What is the fidelity of the input and output states?

5. *A five qubit code.*

   Consider the five qubit stabilizer code given by the Pauli group elements

   $$\langle XZZX\mathbb{1},\ \mathbb{1}XZZX,\ X\mathbb{1}XZZ,\ ZX\mathbb{1}XZ\rangle.$$

   (a) [1 point] Show that this code has one logical qubit.

   (b) [3 points] Show that this code can correct any single qubit Pauli error.