Introduction to quantum information processing

Fault tolerance

Brad Lackey

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OUTLINE

1. Fault tolerant syndrome measurement
2. Shor's protocol for syndrome extraction
3. Code concatenation and the threshold

LAST TIME...

- The Steane code has stabilizer group

$$S = \langle X_1X_3X_5X_7,\ X_2X_3X_6X_7,\ X_4X_5X_6X_7,\ Z_1Z_3Z_5Z_7,\ Z_2Z_3Z_6Z_7,\ Z_4Z_5Z_6Z_7 \rangle.$$

- Explicitly it is

$$\begin{aligned} |0_L\rangle &\mapsto \frac{1}{\sqrt{8}}(|0000000\rangle + |0001111\rangle + |0110011\rangle + |0111100\rangle \\ &\quad + |1010101\rangle + |1011010\rangle + |1100110\rangle + |1101001\rangle). \\ |1_L\rangle &\mapsto \frac{1}{\sqrt{8}}(|1111111\rangle + |1110000\rangle + |1001100\rangle + |1000011\rangle \\ &\quad + |0101010\rangle + |0100101\rangle + |0011001\rangle + |0010110\rangle). \end{aligned}$$

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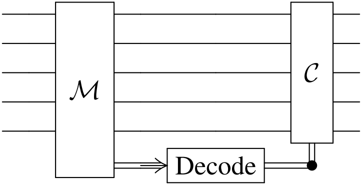
SYNDROME MEASUREMENT

We presented error-correction:

- non-destructive measurement ($\mathcal{M}$),
- then classical syndrome decoding,
- then correction (channel $\mathcal{C}$).



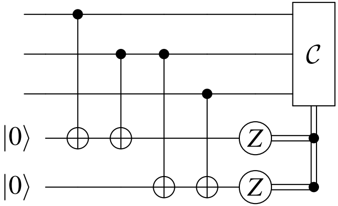
But what if we have errors in our measurement/correction process!

- This act directly on our qubits, so can destroy our system.

Use ancilla to store our syndromes:

- compute parity checks directly,
- measure ancilla (destructively),
- decode syndromes as usual,
- and perform correction.

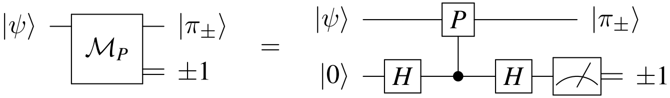
We don't measure our data qubits.



MEASURING SYNDROMES OF STABILIZER CODES

For stabilizer codes, syndromes eigenvalues of Pauli operators.

- For a Hermitian operator P with eigenvalues ± 1 we have:

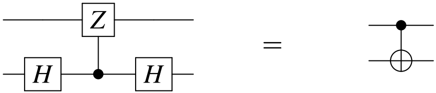


- Why is this: just compute

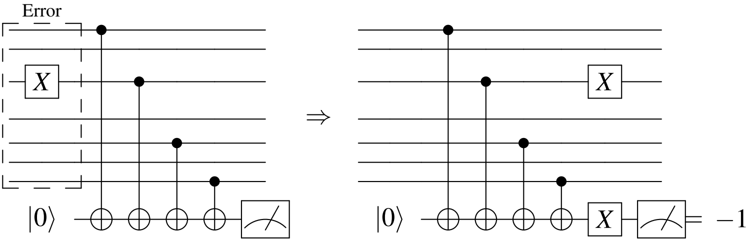
$$|\psi\rangle \otimes |0\rangle \mapsto \tfrac{1}{2}(\mathbb{1} + P)|\psi\rangle \otimes |0\rangle + \tfrac{1}{2}(\mathbb{1} - P)|\psi\rangle \otimes |0\rangle.$$

  Now $\frac{1}{2}(\mathbb{1} \pm P)$ is exactly the projection onto the ± 1 eigenspace.

- This latter circuit often simplifies further. For example:

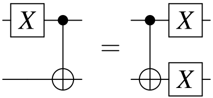


Example computation: Steane code



Consider how the stabilizer $Z_1 Z_3 Z_5 Z_7$ detects a X-error.

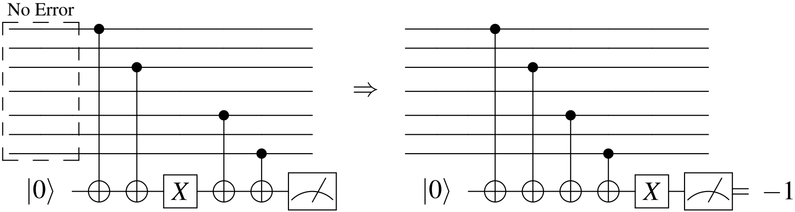
- Envision the error has occurred.
- Using a simple circuit identity we move it.
- Then the measurement detects the error.



Now there's an issue: we made our quantum circuit larger.

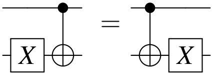
- What if the error is on the syndrome measurement wire.

Example computation: Steane code



If the error is an X-error, it's easy to deal with.

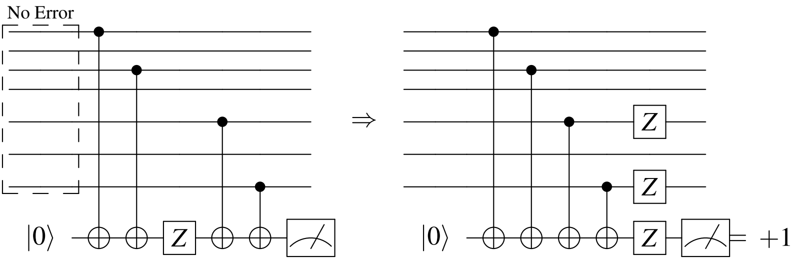
- Envision the error has occurred.
- Using a simple circuit identity we move it.
- Now the error is only in the measurement.



Fix: just repeat this circuit 2 more times.

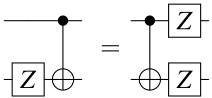
- If there's only 1 error overall, the next two syndromes will be right.

Example computation: Steane code



But if the error is a Z-error, we're in trouble.

- Envision the error has occurred.
- Using a simple circuit identity we move it.
- Now we've got a bit problem.



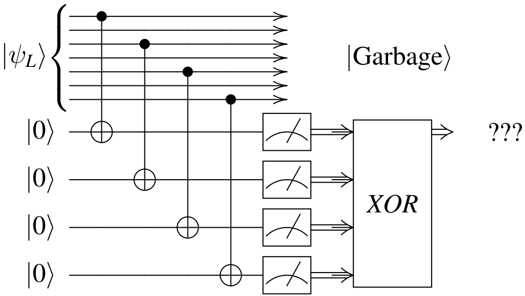
We got the correct syndrome for our data.

- However we've introduced *two* errors, which is uncorrectable.

Outline

Decomposing a syndrome measurement

The problem is that our syndrome computing CNOTs used the same wire.



We might try to split our CNOT computations and classically reassemble.

- But the CNOTs entangle the data state with the ancillae.
- Measuring them collapses the data state, destroying our information.

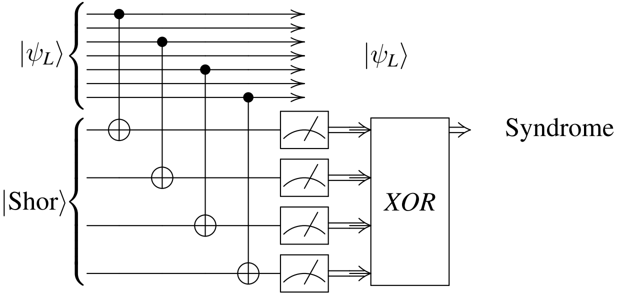
Decomposing a syndrome measurement

Shor devised a interesting trick to deal with:

- use a cleverly designed ancilla $|\text{Shor}\rangle = \frac{1}{\sqrt{8}}\left(\sum_{|v| \text{ even}} |v\rangle\right)$.
- while *each* CNOT entangles, *all* of them do not:

$$\text{CNOT}^{\otimes 4}_{1,3,5,7}|\psi_L\rangle \otimes |\text{Shor}\rangle = |\psi_L\rangle \otimes |\text{Shor}\rangle.$$

- Then, e.g., $\text{CNOT}^{\otimes 4} X_3|\psi_L\rangle \otimes |\text{Shor}\rangle = \frac{1}{\sqrt{8}} X_3|\psi_L\rangle \otimes \sum_{|v| \text{ odd}} |v\rangle$.



Think about: what if the error appears while trying to prepare $|\text{Shor}\rangle$?

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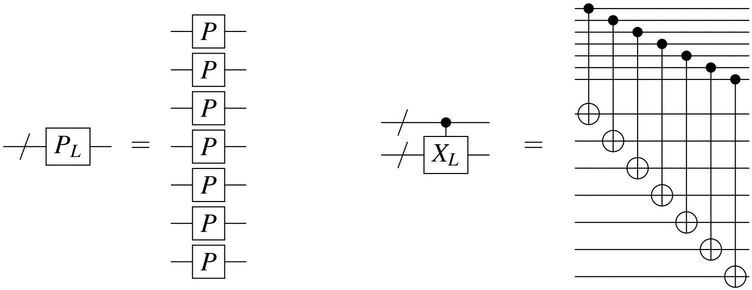
Other fault tolerant operations

In this presentation we focused on syndrome measurement.

- Acting on encode information is critically important too.
- The Steane code is nice because it has many “transversal” operations:



for $P = X, Y, Z$ or even $P = H$.

Threshold

To perform fault-tolerant error correction:

- we added a number ancillae qubits,
- we performed multiple gates between data and ancilla qubits,
- we did multiple measurement and corrections.

This is a very large circuit, and so there are more opportunities for an error.

- Since we can fix any one error, the error rate goes $p \mapsto O(p^2)$.
- But the constant in the $O(\cdot)$ is important. Say $p \mapsto cp^2$.

If $p > \frac{1}{c}$ then $p < cp^2$, so error correction makes things worse!

- When $p < \frac{1}{c}$ error correction improves fidelity.
- The value $\frac{1}{c}$ is called the (pseudo-)threshold of the scheme.

The value c depend heavily on the assumptions about how errors arise.

- Several years ago, I worked out a very simple error model and got $c \approx 1700$ for the Steane code (unoptimized). So $p < 6 \cdot 10^{-4}$.

CODE CONCATENTATION

Note that the tools need to do error correction:

- Pauli and Hadamard gates, CNOTs, and measurement in the Z-bases,
- these can all be done fault-tolerantly.

So we can recurse error correction, and encode the qubits of the code.

- E.g. each of the 7 qubits of the Steane code are logical qubits of a code.
- If we recurse the Steane code k-levels:
  - We use 7^k qubits, but errors reduce $p \mapsto O(p^{2^k})$.

This idea of "code concatenation" leads to the threshold theorem:

- If p is below threshold, then we can reduce errors exponentially with polynomial overhead.

E.g. for $p \mapsto O(p^n)$ with the Steane code, we need $n^{\log_2 7}$ qubits to encode.

Next time...

- Entropy of quantum states.
- Schumacher compression.