Quantum nonlocality

- Nonlocal games
- Bell inequalities

QM exhibits superclassical correlations
this is closely related to operational tasks (teleportation / dense coding, communication, crypto, comm. complexity)

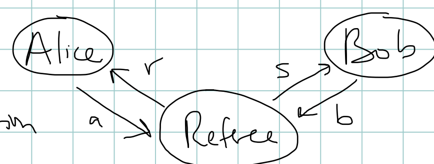
Nonlocal games

useful setting for understanding / quantifying nonlocality

game involves two or more <u>players</u> (Alice, Bob, Charlie, ...)
and a <u>referee</u>

all communication goes through referee
referee asks a question of each player;
players respond with answers
referee evaluates some predicate to decide if players win or lose

Ex: with two players,



(r, s) chosen from some distribution
predicate $P(r, s, a, b)$

A&B can agree on a strategy beforehand, which includes sharing correlations
classical strategies: A&B share correlated random variables
quantum strategies: A&B share entanglement
q. strategies are at least as powerful; sometimes they are more powerful!

Ex: GHZ game

3 players
referee chooses $rst \in_R \{000, 011, 101, 110\}$
sends r to Alice, s to Bob, t to Charlie; they answer with bits a, b, c
they win iff $a \oplus b \oplus c = r \vee s \vee t$:

r	s	t	$a\oplus b\oplus c$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1

best classical strategy:
- first consider deterministic strategy, specified by a_r, b_s, c_t

  to win,
  $$\begin{aligned} a_0 \oplus b_0 \oplus c_0 &= 0 \\ a_0 \oplus b_1 \oplus c_1 &= 1 \\ a_1 \oplus b_0 \oplus c_1 &= 1 \\ a_1 \oplus b_1 \oplus c_0 &= 1 \\ \hline 0 &= 1, \text{ contradiction} \end{aligned}$$

  $\Rightarrow$ they can't always win $\Rightarrow$ success probability $\le \frac{3}{4}$
  (this is achievable: always answer 1)
- randomized strategies can do no better since they give an average of deterministic strategies, and average $\le$ max.

quantum strategy:
suppose A&B share $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
if $r=0$, A measures in $|\pm\rangle$ basis;
if $r=1$, A measures in $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$ basis
(similarly for B & C)

equivalently, if $r=0$, apply $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$;
if $r=1$, apply $K = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ $\qquad K|+i\rangle = |0\rangle,\ K|-i\rangle = |1\rangle$
$\qquad K|0\rangle = |+\rangle,\ K|1\rangle = -i|-\rangle$

rst=000: $H^{\otimes 3}|GHZ\rangle = \frac{1}{4}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle$
$+ |000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle)$
$= \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$, even parity

rst=011: $(H \otimes K \otimes K)|GHZ\rangle = \frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle)$
$= \frac{1}{4}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle$
$- |000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle)$
$= \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle)$, odd parity

Symmetric argument for $rst = 101, 110$.

So there is q. strategy that always succeeds!

In fact, we can get a q. advantage even with only two parties

Ex (CHSH game):

referee chooses $rs \in_R \{00, 01, 10, 11\}$

A & B output bits; they win if $a \oplus b = r \wedge s$

r	s	$a \oplus b$
0	0	0
0	1	0
1	0	0
1	1	1

$a_0 \oplus b_0 = 0$
$a_0 \oplus b_1 = 0$
$a_1 \oplus b_0 = 0$
$a_1 \oplus b_1 = 1$
$0 = 1$

similarly to GHZ game, max. classical success probability is $\frac{3}{4}$

quantum strategy: A & B share a Bell pair,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

given input rs, Alice measures A_r & B_s, where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad B_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

eigenvalues are ± 1; answer 0 for $+1$, 1 for -1

i.e. measure in eigenbases:

A_0: $|0\rangle, |1\rangle$
A_1: $|+\rangle, |-\rangle$
B_0: $\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$, $\sin\frac{\pi}{8}|0\rangle - \cos\frac{\pi}{8}|1\rangle$
B_1: $\cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle$, $\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle$

direct calculation: success probability $\cos^2\frac{\pi}{8}$ for any rs
≈ 0.85

easier to show using observables:

$\langle\psi| A_r \otimes B_s |\psi\rangle = \mathbb{E}[\text{product of A\&B outcomes}]$

so $\Pr(\text{win}) - \Pr(\text{lose}) = \frac{1}{4}\langle\psi| A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 |\psi\rangle$

$$\langle\psi| A_0 \otimes B_0 |\psi\rangle = \frac{1}{2\sqrt{2}}(1\ 0\ 0\ 1)\begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & -1 & -1 \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2}}(1\ 0\ 0\ 1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

and similarly $\langle\psi| A_0 \otimes B_1 |\psi\rangle = \langle\psi| A_1 \otimes B_0 |\psi\rangle = -\langle\psi| A_1 \otimes B_1 |\psi\rangle = 1/\sqrt{2}$

$\Rightarrow \Pr(\text{win}) - \Pr(\text{lose}) = \frac{1}{\sqrt{2}}$

$= \Pr(\text{win}) - (1 - \Pr(\text{win}))$

$= 2\Pr(\text{win}) - 1$

$\Rightarrow \Pr(\text{win}) = \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2 \frac{\pi}{8} \approx 0.85 > \frac{3}{4}$

Bell inequalities

This example was originally cast in terms of a Bell inequality, an inequality satisfied by any local hidden variable theory (shared random variables of A & B that they can use to determine measurement outcomes) but violated by QM

The inequality is $A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$

If A_0, A_1, B_0, B_1 are classical variables, this inequality must hold in $[-1, 1]$

But $\langle \Psi | A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 | \Psi \rangle = 2\sqrt{2}$, a violation of the Bell inequality

experiments confirm this!

[recent work simultaneously closes "locality loophole" (A & B can't communicate) & "sampling loophole" (can't explain violation by failure to detect bad events)]

Tsirelson's bound: $\langle \Psi | A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 | \Psi \rangle \leq 2\sqrt{2}$

so the above violation is the strongest possible.