Introduction to quantum information processing

Partial Trace and State Purification

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27 October 2016

Outline

Last time...

We found quantum states were more general than first thought.

- States of the form $|\psi\rangle$ are “pure” states.
- Ensembles and states with external information are “mixed” states.
- Mixed states are density operators ρ.
- Pure states can be written as a mixed state via $\rho = |\psi\rangle\langle\psi|$.

Outline

Projection-valued measures

Projective, or *orthogonal*, measurements are derived from observables.

- Basic idea: in the lab an observation produces an “expected value.”
- An observable is represented by a Hermitian operator.
- Observing A when the system is in state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|A|\psi\rangle$.

Let’s use the spectral decomposition $A = \sum_{j=1}^{r} \lambda_j |\phi_j\rangle\langle\phi_j|$ to expand this:

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_{j=1}^{r} \lambda_j \langle\psi|\Pi_j|\psi\rangle.$$

- We have a probability of the event j happening: $\langle\psi|\Pi_j|\psi\rangle$.
- The eigenvalue λ_j is the number that gets observed in the experiment.
- Then $\langle A\rangle$ is just the mathematical formula for “expectation.”

The spectral projections $\{\Pi_j\}_{j=1}^{r}$ define our “measurement basis.”

Gleason's theorem

What should a state satisfy?

- It should give us probabilities. So if Π is a projection then $p(\Pi) \in [0, 1]$.
- If $\{\Pi_j\}_{j=1}^N$ is a projection-valued measure, then $\sum_{j=1}^N p(\Pi_j) = 1$.

So can there be even things beyond densities that could be "states?"

- Yes and no. Gleason Theorem addresses this precisely.
- If $\dim \mathfrak{H} = 2$, then some funny things can happen.
- Otherwise density operators are precisely the set of states.

Theorem (Gleason (1957))

Suppose $\dim \mathfrak{H} \geq 3$. *Let* $p : \mathcal{P}(\mathfrak{H}) \to [0, 1]$ *be any function satisfying* $\sum_{j=1}^N p(\Pi_j) = 1$ *whenever* $\{\Pi_j\}_{j=1}^N$ *is a projection-valued measure. Then there exists a unique density operator* ρ *such that* $p(\Pi) = \mathrm{tr}(\Pi\rho)$.

The rule $\Pr_\rho(\Pi) = \mathrm{tr}(\Pi\rho)$ is called *Born's rule*.

Positive operator-valued measures

Or simply POVMs

Can we have more than two outcomes for a qubit measurement?

- No! We can only have two (nonzero) orthogonal vectors in 2-d.

A "generalized" measurement, or *positive operator-valued measure* is:

- A collection of positive operators $\{E_j\}_{j=1}^{m}$ (i.e. $E_j \geq 0$),
- that have sum $\sum_{j=1}^{m} E_j = \mathbb{1}$.

We define the POVM probability rule to mimic Born's rule:

$$\Pr_\rho\{E_j\} = \mathrm{tr}(E_j \rho).$$

A common example is where the outcomes might be $\{0, 1, \perp\}$.

- Here $\{0, 1\}$ could be a bit or a no/yes response, and $\perp$ is "fail."
- Problem 4.6 deals with POVMs $\{E_0, E_1, E_\perp\}$ having a fail option.

Outline

1. Measurements
2. Schmidt decomposition
3. Marginals
4. State purification

Bipartite systems

Some of the most interesting science comes from multipartite systems.

- We describe bipartite systems via two actors Alice and Bob.
- Today: Alice will be "our local system" and Bob "an external system."
- Later: Alice will send (quantum) information, and Bob receives it.

If Alice and Bob are completely independent:

- their joint probabilities would be

$$\Pr\{A = a \text{ and } B = b\} = \Pr\{A = a\} \cdot \Pr\{B = b\}.$$

- in the quantum case this can be achieve by $\rho_A \otimes \rho_B$.
- In particular, bipartite systems are composed as $\mathfrak{H}_A \otimes \mathfrak{H}_B$.

The Schmidt decomposition

Theorem (Schmidt decomposition)

Let $|\psi\rangle \in \mathfrak{H}_A \otimes \mathfrak{H}_B$. Then there exists orthonormal basis $\{|\phi_j^{(A)}\rangle\}$ and $\{|\phi_\mu^{(B)}\rangle\}$ of $\mathfrak{H}_A$ and $\mathfrak{H}_B$ and positive values $\{\lambda_j\}$ so that

$$|\psi\rangle = \sum_{j=1}^{r} \sqrt{\lambda_j} |\phi_j^{(A)}\rangle \otimes |\phi_j^{(B)}\rangle.$$

- $r \leq \min\{\dim \mathfrak{H}_A, \dim \mathfrak{H}_B\}$ is the “Schmidt number” (or “rank”) of $|\psi\rangle$.
- The “Schmidt coefficients” are the $\sqrt{\lambda_j}$.

The Schmidt decomposition is basically the singular value decomposition.

1. Take any bases of $\mathfrak{H}_A$ and $\mathfrak{H}_B$ and write $|\psi\rangle = \sum_{j\mu} \alpha_{j\mu} |\chi_j^{(A)}\rangle \otimes |\chi_\mu^{(B)}\rangle$.
2. Use the singular value decomposition: $\alpha = UDV$ where

$$D = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}, 0, \ldots).$$

3. Form $\{|\phi_j^{(A)}\rangle\}$ and $\{|\phi_\mu^{(B)}\rangle\}$ using $U^\dagger$ and V as change of basis.

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NOTION OF MARGINALS IN PROBABILITY

For a bipartite system a marginal (probability or distribution) refers to:

- probabilities associated with one part of the system,
- without reference to the other part of the system.

This is different that *conditional* probability where the outcome in the other part is known and accounted for.

For classical probability the formula reads:

$$p(a) = \Pr\{A = a\} = \sum_b \Pr\{A = a \text{ and } B = b\} = \sum_b p(a,b).$$

In the quantum case:

- the joint probability is give by a density ρ on $\mathfrak{H}_A \otimes \mathfrak{H}_B$,
- Alice's marginal probability should be given by a density ρ' on $\mathfrak{H}_A$.
- In the above classical formula, we "sum over Bob's outcomes."

Partial trace

In the quantum setting, "summing over Bob's state" is called "partial trace."

We can formalize it via the scenario:

- Alice wants to make a measurement, but Bob's not so interested.
- So measurement projections will be of the form $\Pi \otimes \mathbb{1}_B$.
- The resulting probabilities (w.r.t. Alice) are $\operatorname{tr}((\Pi \otimes \mathbb{1}_B)\rho)$.
- By Gleason's theorem, there exists a density ρ' with

$$\operatorname{tr}(\Pi\rho') = \operatorname{tr}((\Pi \otimes \mathbb{1}_B)\rho).$$

This ρ' is the *partial trace* (over Bob's subsystem).

- We denote this as $\rho' = \operatorname{tr}_B(\rho)$.
- *Warning*: we write tr_B but this is an operator on $\mathfrak{H}_A$!
- It is the quantum analogue of Alice's marginal probability.

Computing the partial trace

To compute: take orthonormal basis $\{|\phi_j^{(A)}\rangle\}$ and $\{|\phi_\mu^{(B)}\rangle\}$ of $\mathfrak{H}_A$ and $\mathfrak{H}_B$.

$$\begin{aligned}
\mathrm{tr}((\Pi \otimes \mathbb{1}_B) \otimes \rho) &= \sum_{j\mu} (\langle \phi_j^{(A)}| \otimes \langle \phi_\mu^{(B)}|)((\Pi \otimes \mathbb{1}_B)\rho)(|\phi_j^{(A)}\rangle \otimes |\phi_\mu^{(B)}\rangle) \\
&= \sum_{j\mu} \langle \phi_j^{(A)}| \left(\Pi \cdot \langle \phi_\mu^{(B)}|\rho|\phi_\mu^{(B)}\rangle \right) |\phi_j^{(A)}\rangle \\
&= \sum_{j} \langle \phi_j^{(A)}| \left(\Pi \cdot \sum_\mu \langle \phi_\mu^{(B)}|\rho|\phi_\mu^{(B)}\rangle \right) |\phi_j^{(A)}\rangle
\end{aligned}$$

And so we see why this is called the “partial” trace:

$$\mathrm{tr}_B(\rho) = \sum_\mu \langle \phi_\mu^{(B)}|\rho|\phi_\mu^{(B)}\rangle.$$

A specific example: $\rho = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$ Then

$$\begin{aligned}
\mathrm{tr}_B(\rho) &= \frac{1}{2}\left(\langle 0^{(B)}|\rho|0^{(B)}\rangle + \langle 1^{(B)}|\rho|1^{(B)}\rangle \right) \\
&= \frac{1}{2}\left(|0^{(A)}\rangle\langle 0^{(A)}| + |1^{(A)}\rangle\langle 1^{(A)}| \right) = \frac{1}{2}\mathbb{1}_A.
\end{aligned}$$

Outline

1 Measurements

2 Schmidt decomposition

3 Marginals

4 State purification

Marginals of pure states

Theorem

Let $|\psi\rangle \in \mathfrak{H}_A \otimes \mathfrak{H}_B$ be a pure state. Then the marginals $\mathrm{tr}_A(|\psi\rangle\langle\psi|)$ and $\mathrm{tr}_B(|\psi\rangle\langle\psi|)$ have the same nonzero eigenvalues.

Proof. Let $|\psi\rangle = \sum_j \sqrt{\lambda_j}|\phi_j^{(A)}\rangle \otimes |\phi_j^{(B)}\rangle$ be the Schmidt decomposition. Then:

$$|\psi\rangle\langle\psi| = \sum_{jk} \sqrt{\lambda_j\lambda_k}|\phi_j^{(A)}\rangle\langle\phi_k^{(A)}| \otimes |\phi_j^{(B)}\rangle\langle\phi_k^{(B)}|.$$

Using the orthonormal basis $\{|\phi_\mu^{(B)}\rangle\}$ to compute the partial trace:

$$\begin{aligned}\mathrm{tr}_B(|\psi\rangle\langle\psi|) &= \sum_\mu \langle\phi_\mu^{(B)}|(|\psi\rangle\langle\psi|)|\phi_\mu^{(B)}\rangle \\ &= \sum_{jk} \sqrt{\lambda_j\lambda_k}|\phi_j^{(A)}\rangle\langle\phi_k^{(A)}| \sum_\mu \langle\phi_\mu^{(B)}|\phi_j^{(B)}\rangle\langle\phi_k^{(B)}|\phi_\mu^{(B)}\rangle \\ &= \sum_j \lambda_j|\phi_j^{(A)}\rangle\langle\phi_j^{(A)}|.\end{aligned}$$

This form is diagonal, so λ_j are the nonzero eigenvalues. Same for tr_A. □

State purification

A fascinating result is that this theorem can be reversed:

- If Alice has mixed state ρ, she can realize it as $\rho = \operatorname{tr}_B(|\psi\rangle\langle\psi|)$.
- I.e. including external (Bob's) information, $|\psi\rangle \in \mathfrak{H}_A \otimes \mathfrak{H}_B$ is pure.

Here's how it works:

1. Use the spectral theorem to write $\rho = \sum_{j=1}^{r} \lambda_j |\phi_j\rangle\langle\phi_j|$.
2. Declare $\{|\chi_j\rangle\}_{j=1}^{r}$ orthonormal for a Hilbert space $\mathfrak{H}_B = \operatorname{span}\{|\chi_j\rangle\}$.
3. Define $|\psi\rangle = \sum_j \sqrt{\lambda_j} |\phi_j\rangle \otimes |\chi_j\rangle \in \mathfrak{H}_A \otimes \mathfrak{H}_B$.
4. The previous computation shows $\rho = \operatorname{tr}_B(|\psi\rangle\langle\psi|)$.

In words, given an ensemble $\{(|\phi_j\rangle, \lambda_j)\}_{j=1}^{r}$:

- each species gets associated to a new quantum state (via the $|\chi_j\rangle$),
- and attaching this information to the prepared $|\phi_j\rangle$ purifies ρ.

There's some question about using coefficient $\sqrt{\lambda_j}$, but the math sorts it out.

Next time...

- Superoperators.
- Kraus operators.
- Examples of some quantum channels.
- Stinespring dilation theorem.